Homework Set # 13

19.5.4

$$V(r) = -V_0 \, \Theta(r_0 - r)$$

For the s-wave ($l = 0$)

$$\psi = \psi(r) = A\, j_0(kr) + B\, \eta_0(kr)$$

$$= \frac{1}{kr}(A \sin kr - B \cos kr) \qquad \text{for } r > r_0$$

with $E = \frac{\hbar^2}{2\mu} k^2$, and

$$\psi(r) = C\, j_0(k'r) = \frac{C}{k'r} \sin k'r \qquad \text{for } r < r_0$$

with $E = \frac{\hbar^2 k'^2}{2\mu} - V_0$.

$\psi(r)$ and $\frac{d\psi}{dr}$ must be continuous at $r = r_0$.

Hence

$$\frac{1}{k'} C \sin k' r_0 = \frac{1}{k}(A \sin k r_0 - B \cos k r_0)$$

and $$C \cos k' r_0 = A \cos k r_0 + B \sin k r_0$$

$$A = \left(\frac{k}{k'} \sin k'r_0 \sin kr_0 + \cos k'r_0 \cos kr_0 \right) C$$

$$B = \left(-\frac{k}{k'} \sin k'r_0 \cos kr_0 + \cos k'r_0 \sin kr_0 \right) C$$

The phase shift δ_ℓ is defined by

$$\psi_{\vec{k}}(\vec{r}) \xrightarrow[r \to \infty]{} \sum_{l=0}^{\infty} A_\ell \frac{e^{i(kr - l\frac{\pi}{2} + \delta_\ell)} - e^{-i(kr - l\frac{\pi}{2} + \delta_\ell)}}{r} P_\ell(\cos\theta)$$

In our case ($l = 0$)

$$\psi(r) = \frac{1}{ikr}\left[(A - iB) e^{ikr} - (A + iB) e^{-ikr} \right]$$

for $r > r_0$

Hence

$$e^{2i\delta_0} = \frac{A - iB}{A + iB} = \frac{\cos k'r_0 \, e^{-ikr_0} + i \frac{k}{k'} \sin k'r_0 \, e^{-ikr_0}}{\cos k'r_0 \, e^{ikr_0} - i \frac{k}{k'} \sin k'r_0 \, e^{ikr_0}}$$

$$= e^{-2ikr_0} \frac{\cos k'r_0 + i \frac{k}{k'} \sin k'r}{\cos k'r_0 - i \frac{k}{k'} \sin k'r}$$

$$= e^{-2ikr_0} \, e^{2i\beta}$$ where

$$\tan\beta = \frac{k}{k'}\tan k' r_0$$

$$\therefore \quad \delta_0 = -k r_0 + \tan^{-1}\left(\frac{k}{k'}\tan k' r_0\right)$$

For $\quad k' r_0 = (n+\tfrac{1}{2})\pi + \underset{\text{small}}{\underset{\uparrow}{\epsilon}}$

$$\frac{k}{k'}\tan k' r_0 \simeq -\frac{k}{k'}\frac{1}{\epsilon}$$

$$\simeq \frac{-k}{k'\left[k' r_0 - (n+\tfrac{1}{2})\pi\right]}$$

$$\simeq \frac{-\frac{k\hbar^2}{2\mu}\frac{1}{r_0}}{\frac{\hbar^2 k'^2}{2\mu} - V_0 + V_0 - \frac{\hbar^2 k'}{2\mu r_0}(n+\tfrac{1}{2})\pi}$$

$$\simeq \frac{\Gamma/2}{V_0 - \frac{\hbar^2}{2\mu r_0^2}(n+\tfrac{1}{2})^2\pi^2 - E}$$

with $\quad \Gamma = \dfrac{\hbar^2 k_n}{\mu r_0}$

From Exercise 12.6.9 , for bound states

$$k'/\chi = -\tan k' r_0$$

where $\chi = \frac{1}{\hbar}\sqrt{-2\mu E}$

For zero energy bound states, $\chi \to 0$

$$\tan k'_n r_0 = \infty$$

$$\therefore \quad k'_n r_0 = (n+\tfrac{1}{2})\pi$$

$$n = 0, 1, 2 \ldots$$

19.5.6

i.

$$\psi = e^{ikz} + f(\theta)\frac{e^{ikr}}{r} \quad \text{as } r \to \infty$$

$$\vec{j}^{\,int} = \frac{\hbar}{2\mu i}\left(\psi^* \vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$$

$$= \frac{\hbar}{\mu}\,\mathrm{Im}\left[e^{-ikz}\,\vec{\nabla}\left(f(\theta)\frac{e^{ikr}}{r}\right) + f^*(\theta)\frac{e^{-ikr}}{r}\,\vec{\nabla}e^{ikz}\right]$$

$$j_r^{int} = \frac{\hbar}{\mu}\,\mathrm{Im}\,\hat{r}\cdot\left[e^{-ikz}f(\theta)\frac{e^{ikr}}{r}\left(ik-\frac{1}{r}\right) + f^*(\theta)\frac{e^{-ikr+ikz}}{r}\,ik\hat{z}\right]$$

$$= \frac{\hbar k}{\mu r}\,\mathrm{Im}\Big[e^{+ikr(1-\cos\theta)}f(\theta)\,i$$
$$+ f^*(\theta)\,i\,e^{ikr(\cos\theta-1)}\cos\theta\Big]$$
$$+O\left(\frac{1}{r^2}\right)$$

ii.

$$I_{\Delta\Omega} = \int_{\Delta\Omega} r^2 d\Omega\, j_r^{int} \qquad \Delta\Omega = \Delta\varphi\sin\theta\Delta\theta$$
$$= \Delta\varphi\,\Delta\cos\theta$$
$$= \Delta\varphi\,\Delta x$$

$$= \frac{\hbar^2 k}{\mu r}\,\mathrm{Im}\Big\{i f(\theta)\,\Delta\varphi\int_x^{x+\Delta x} dx\, e^{ikr(1-x)}$$

$$+ i f^*(\theta)\,\Delta\varphi\int_x^{x+\Delta x} dx\, e^{ikr(x-1)}\,x\Big\}$$

$$I^{int}_{\Delta\Omega} = \frac{\hbar\,\Delta\varphi}{\mu}\, \mathrm{Im}\left\{ f(\theta)\left[e^{ikr(1-\cos(\theta+\Delta\theta))} - e^{ikr(1-\cos\theta)} \right] \right.$$

$$\left. - f^*(\theta)\cos\theta \left[e^{-ikr(1-\cos(\theta+\Delta\theta))} - e^{-ikr(1-\cos\theta)} \right] \right\}$$

For $\theta \neq 0$ and $\Delta\theta$ small,

$I^{int}_{\Delta\Omega}$ average to zero as $r \to \infty$.

iii. Let $\theta = 0$, $\Delta\theta$ small and $\Delta\varphi = 2\pi$.

$$I^{int}_{\substack{\Delta\Omega \\ \text{forward}}} = \frac{\hbar\, 2\pi}{\mu}\, \mathrm{Im}\left\{ f(0)\left[e^{ikr(1-\cos\Delta\theta)} - 1 \right] \right.$$

$$\left. - f^*(0)\left[e^{-ikr(1-\cos\Delta\theta)} - 1 \right] \right\}$$

The exponentials average to zero as $r \to \infty$.

Hence

$$I^{int}_{\substack{\Delta\Omega \\ \text{forward}}} = -\frac{\hbar\, 2\pi}{\mu}\, \mathrm{Im}\left(f(0) - f^*(0) \right)$$

$$I^{int}_{\substack{\Delta\Omega \\ \text{forward}}} = -\frac{4\pi\hbar}{\mu} \operatorname{Im} f(0)$$

Note that the RHS does not depend upon $\Delta\Omega = \pi(\Delta\theta)^2$.

$$\vec{j}^{\,tot} = \frac{\hbar}{\mu} \operatorname{Im}\left[(\psi^{inc} + \psi^{scatt})^* \vec{\nabla}(\psi^{inc} + \psi^{scatt}) \right]$$

$$= \frac{\hbar k}{\mu}\hat{z} + \vec{j}^{\,int} + \frac{\hbar}{\mu} \operatorname{Im}\left(\psi^{scatt\,*} \vec{\nabla} \psi^{scatt}\right)$$

$$I^{tot}_{\substack{\Delta\Omega \\ \text{forward}}} = \frac{\hbar k}{\mu} r^2 \Delta\Omega + I^{int}_{\substack{\Delta\Omega \\ \text{forward}}}$$

$$+ \frac{\hbar}{\mu} |f(0)|^2 k \, \Delta\Omega$$

Take the limit

$$r \to \infty , \quad \Delta\Omega \to 0$$

$$\Delta S = r^2 \Delta\Omega \quad \text{finite}$$

The last term drops out.

In the absence of scatterer

$$I^{int}_{\substack{\Delta S \\ forward}} = \frac{\hbar k}{\mu} \Delta S = (\text{incident flux}) \times \Delta S$$

When the scatterer is present

$$I^{int}_{\substack{\Delta S \\ forward}} = \frac{\hbar k}{\mu} \Delta S - \frac{4\pi \hbar}{\mu} \operatorname{Im} f(0)$$

Hence

$$\frac{4\pi \hbar}{\mu} \operatorname{Im} f(0) = \#\text{ particles scattered per unit time}$$

$$= \sigma_{tot} \times \text{incident flux}$$

$$= \sigma_{tot} \frac{\hbar k}{\mu}$$

Hence

$$\sigma_{tot} = \frac{4\pi}{k} \operatorname{Im} f(0).$$

S19.6.1

$$\frac{d\sigma}{d\Omega_L} d\Omega_L = \frac{d\sigma}{d\Omega} d\Omega$$

$$\left.\frac{d\sigma}{d\Omega}\right|_L = |f(\theta_L, \varphi_L)|^2$$

i. In equal mass case $\theta_L = \frac{\theta}{2}$

$$\left.\frac{d\sigma}{d\Omega_L}\right|_{\theta_L} = \left.\frac{d\sigma}{d\Omega}\right|_{\theta=2\theta_L} \frac{d\Omega}{d\Omega_L}$$

$$\begin{aligned} d\Omega &= d\varphi \sin\theta \, d\theta = d\varphi \, |d\cos\theta| \\ &= d\varphi_L |d\cos(2\theta_L)| = d\varphi_L \sin 2\theta_L \, 2 \, d\theta_L \\ &= 4 \, d\varphi_L \sin\theta_L \, d\theta_L \cos\theta_L \\ &= 4 \cos\theta_L \, d\Omega_L \end{aligned}$$

$$\therefore \quad \left.\frac{d\sigma}{d\Omega_L}\right|_{\theta_L} = \left.\frac{d\sigma}{d\Omega}\right|_{\theta=2\theta_L} 4\cos\theta_L$$

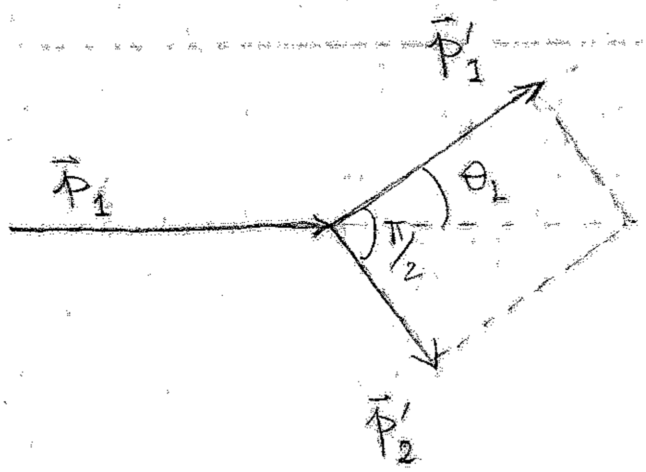
ii. $$E = \underbrace{\frac{1}{2m_1} \vec{p}_1 \cdot \vec{p}_1}_{\text{initial energy}} = \underbrace{\frac{1}{2m_1} \vec{p}_1{}' \cdot \vec{p}_1{}' + \frac{1}{2m_2} \vec{p}_2{}' \cdot \vec{p}_2{}'}_{\text{final energy}} \qquad (\vec{p}_2 = 0)$$

$$\vec{P} = \vec{p}_1 = \vec{p}'_1 + \vec{p}'_2$$

$$E = \frac{1}{2m_1}\vec{P}_1 \cdot \vec{P}_1 = \frac{1}{2m_1}\left[\vec{P}'_1 \cdot \vec{P}'_1 + 2\vec{P}'_1 \cdot \vec{P}'_2 + \vec{P}'_2 \cdot \vec{P}'_2\right]$$

$$= \frac{1}{2m_1}\vec{P}'_1 \cdot \vec{P}'_1 + \frac{1}{2m_2}\vec{P}'_2 \cdot \vec{P}'_2$$

If $m_1 = m_2$, $\vec{P}'_1 \cdot \bar{P}'_2 = 0$



If $\theta_L > \pi/2$, $\vec{P}'_1$ and $\bar{P}'_2$ would have to be on the same side of $\bar{P}_1$ which is impossible by (transverse) momentum conservation.

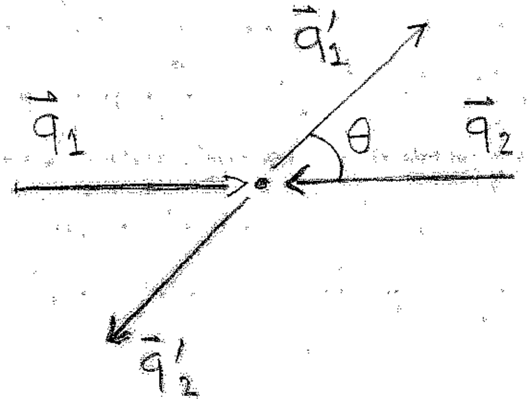
iii. $m_1 \neq m_2$

The velocity of the center of mass is

$$\vec{V}_{CM} = \frac{1}{m_1+m_2} \vec{P}_1$$

The initial CM momenta are

$$\vec{q}_1 = \vec{P}_1 - m_1 \vec{V}_{CM} \quad , \quad \vec{q}_2 = -m_2 \vec{V}_{CM} = -\vec{q}_1$$



Let $\vec{q}_1'$ and $\vec{q}_2'$ be the final CM momenta

We have $\vec{q}_2' = -\vec{q}_1'$ and $|\vec{q}_1'| = |\vec{q}_1| \equiv q$

$$P_{1\perp}' = q_{1\perp}' = q \sin\theta$$

$$P_{1\parallel}' = q_{1\parallel}' + m_1 V_{cM}$$

$$= q \cos\theta + \frac{m_1}{m_1+m_2} P_1$$

But

$$q = |\vec{q}_1| = P_1 - m_1 V_{cM}$$

$$= P_1 - m_1 \frac{P_1}{m_1 + m_2} = \frac{m_2}{m_1 + m_2} P_1$$

$$\therefore \quad P'_{1\parallel} = q\left(\cos\theta + \frac{m_1}{m_2}\right)$$

$$\therefore \quad \tan\theta_L = \frac{P'_{1\perp}}{P'_{1\parallel}} = \frac{\sin\theta}{\cos\theta + \frac{m_1}{m_2}}$$