Homework Set #14

20.1.1

$$i\hbar \frac{\partial \psi}{\partial t} = \left(c\,\vec{\alpha} \cdot \frac{\hbar}{i}\vec{\nabla} + \beta m c^2 \right)\psi$$

$$\frac{\partial \psi}{\partial t} = \left(-c\,\vec{\alpha}\cdot\vec{\nabla} - \frac{i m c^2}{\hbar}\beta \right)\psi$$

$$\frac{\partial \psi^\dagger}{\partial t} = \psi^\dagger \left(-c\,\vec{\alpha}^\dagger \cdot \overleftarrow{\nabla} + \frac{i m c^2}{\hbar}\beta^\dagger \right)$$

$$= \psi^\dagger \left(-c\,\vec{\alpha}\cdot\overleftarrow{\nabla} + \frac{i m c^2}{\hbar}\beta \right)$$

since $\vec{\alpha}^\dagger = \vec{\alpha}$ and $\beta^\dagger = \beta$.

By $\overleftarrow{\nabla}$ we mean that the gradient operator acts on the left

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \qquad \vec{\nabla}\psi = \begin{pmatrix} \vec{\nabla}\psi_1 \\ \vec{\nabla}\psi_2 \\ \vec{\nabla}\psi_3 \\ \vec{\nabla}\psi_4 \end{pmatrix}$$

$$\psi^\dagger = \left(\psi_1^* \quad \psi_2^* \quad \psi_3^* \quad \psi_4^* \right)$$

$$\psi^\dagger \overleftarrow{\nabla} = \left(\vec{\nabla}\psi_1^* \quad \vec{\nabla}\psi_2^* \quad \vec{\nabla}\psi_3^* \quad \vec{\nabla}\psi_4^* \right)$$

$$\frac{\partial}{\partial t}\left(\psi^\dagger \psi\right) = \psi^\dagger \left(-c\vec{\alpha}\cdot\vec{\nabla} - \frac{imc^2}{\hbar}\beta \right)\psi$$

$$+ \psi^\dagger \left(-c\vec{\alpha}\cdot\overleftarrow{\nabla} + \frac{imc^2}{\hbar}\beta \right)\psi$$

$$= -c\left(\psi^\dagger \vec{\alpha}\cdot\vec{\nabla}\psi + \psi^\dagger \vec{\alpha}\cdot\overleftarrow{\nabla}\psi \right)$$

$$= -c\,\vec{\nabla}\cdot\left(\psi^\dagger \vec{\alpha}\psi\right)$$

i.e. $\dfrac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$ with

$\rho = \psi^\dagger\psi$ and

$\vec{j} = c\,\psi^\dagger \vec{\alpha}\psi$.

20.2.1

$$\vec{\pi} \times \vec{\pi} \ \psi$$

$$= \hat{k}\, \epsilon_{j\ell k} \left(\frac{\hbar}{i} \partial_j - \frac{q}{c} A_j \right) \left(\frac{\hbar}{i} \partial_\ell - \frac{q}{c} A_\ell \right) \psi$$

$$= \hat{k} \left(-\frac{\hbar q}{ic} \right) \epsilon_{j\ell k} \left[\partial_j (A_\ell \psi) - A_j \partial_\ell \psi \right]$$

$$= \hat{k} \left(-\frac{\hbar q}{ic} \right) \epsilon_{j\ell k} \left(\partial_j A_\ell \right) \psi$$

$$= -\frac{\hbar q}{ic} B_k \hat{k} \psi$$

$$\therefore \quad \vec{\pi} \times \vec{\pi} = \frac{i\hbar q}{c} \vec{B}$$

20.2.2

$$\left[c\vec{\alpha} \cdot \left(\frac{\hbar}{i} \vec{\nabla} - \frac{q}{c} \vec{A} \right) + \beta m c^2 + q\phi \right] \psi$$

$$= E \psi$$

For $\psi = \begin{pmatrix} \chi \\ \Phi \end{pmatrix}$ and $\phi = 0$,

we have [Eqs. (20.2.7) and (20.2.8)]

$$(E - \mu c^2) \chi = c\, \vec{\sigma} \cdot \vec{\pi}\, \Phi$$

$$(E + \mu c^2) \Phi = c\, \vec{\sigma} \cdot \vec{\pi}\, \chi$$

with $\vec{\pi} = \frac{\hbar}{i} \vec{\nabla} - \frac{q}{c} \vec{A}$

$\therefore$ $$(E^2 - \mu^2 c^4) \chi = c\, \vec{\sigma} \cdot \vec{\pi}\, (E + \mu c^2) \Phi$$

$$= c^2 (\vec{\sigma} \cdot \vec{\pi})^2 \chi$$

$$= c^2 \left(\vec{\pi} \cdot \vec{\pi} - \frac{q\hbar}{c} \vec{\sigma} \cdot \vec{B} \right) \chi$$

using Eqs. (20.2.15) and (20.2.16)

For $\vec{B} = B_0 \hat{z}$

$$\vec{A} = \frac{1}{2} B_0 (-y \hat{x} + x \hat{y})$$

$$\left(E^2 - \mu^2 c^4\right) \chi$$

$$= c^2 \left[-\hbar^2 \frac{\partial^2}{\partial z^2} + \left(\frac{\hbar}{i} \frac{\partial}{\partial x} + \frac{qB_0}{2c} y \right)^2 \right.$$

$$\left. + \left(\frac{\hbar}{i} \frac{\partial}{\partial y} - \frac{qB_0}{2c} x \right)^2 - \frac{q\hbar}{c} \vec{\sigma} \cdot \vec{B} \right] \chi$$

$$= c^2 \left[-\hbar^2 \nabla^2 + \frac{qB_0 \hbar}{ic} \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) \right.$$

$$\left. + \left(\frac{qB_0}{2c} \right)^2 (x^2 + y^2) - \frac{q\hbar B_0}{c} \sigma_z \right] \chi$$

Let

$$\chi = e^{\frac{i}{\hbar} p_z z} \begin{pmatrix} \chi_\uparrow (x,y) \\ \chi_\downarrow (x,y) \end{pmatrix}$$

Then

$$\frac{1}{c^2} \left(E^2 - \mu^2 c^4 - c^2 p_z^2 \right) \chi_{\uparrow\downarrow}$$

$$= \left[-\hbar^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \left(\frac{qB_0}{2c} \right)^2 (x^2 + y^2) \right.$$

$$\left. - \frac{qB_0}{c} \left(L_z \pm \hbar \right) \right] \chi_{\uparrow\downarrow}$$

See the solutions to previous homework problems 12.3.7 and 12.3.8

$$H' = \frac{1}{2\mu}\left(P_x^2 + P_y^2\right) + \frac{1}{2}\mu\omega^2\left(x^2 + y^2\right)$$

is invariant under two dimensional rotations

$$[H', L_z] = 0$$

Its eigenvalues are

$$E' = (2k + |m| + 1)\hbar\omega$$

with $k = 0, 1, 2 \ldots$ and $m = 0, \pm 1, \pm 2, \ldots$

$m\hbar$ is the eigenvalue of L_z.

The eigenvalues of

$$-\hbar^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \left(\frac{qB_0}{2c}\right)^2(x^2 + y^2) - \frac{qB_0}{c}L_z$$

are therefore

$$2\mu\left[(2k + |m| + 1)\hbar\omega - \omega\hbar m\right]$$

with $\omega = \dfrac{qB_0}{2\mu c}$.

Note that this consistent with the answer to problem 12.3.8

$$(2k + |m| - m + 1)\hbar\omega$$

$$= (n + \tfrac{1}{2})\hbar\omega_0 \quad \text{with}$$

$$\omega_0 = \frac{qB_0}{\mu c} = 2\omega \quad \text{and}$$

$$n = k + \tfrac{1}{2}(|m| - m)$$

$$= 0, 1, 2, 3 \ldots$$

In conclusion,

$$E^2 = \mu^2 c^4 + c^2 p_z^2$$

$$+ 2\mu c^2 \left[2k + (|m| - m) + 1 - 2m_s \right] \hbar\omega$$

$$E = \pm \Big[\mu^2 c^4 + c^2 p_z^2$$

$$+ 2\mu c^2 \left[2k + (|m| - m) + 1 - 2m_s \right] \hbar\omega \Big]^{1/2}$$

where $k = 0, 1, 2 \ldots$

$m = 0, \pm 1, \pm 2, \ldots$

$m_s = \pm \frac{1}{2}$

$\hbar m$ is the eigenvalue of L_z

$\hbar m_s$ " " " " S_z.

21.1.16

$$z = x + iy \qquad z^* = x - iy$$

$$x = \frac{1}{2}(z + z^*) \qquad y = \frac{1}{2i}(z - z^*)$$

$$dx\, dy = dz\, dz^* \det \begin{pmatrix} 1/2 & 1/2 \\ 1/2i & -1/2i \end{pmatrix}$$

$$= \frac{1}{2i}\, dz\, dz^*$$

$$\int \frac{dx\, dy}{\pi} |z\rangle\langle z| e^{-z^* z}$$

$$= \int \frac{dz\, dz^*}{2\pi i} |z\rangle\langle z| e^{-z^* z}$$

$$= \int \frac{dz\, dz^*}{2\pi i} \sum_{m,n=0}^{\infty} \frac{z^n}{\sqrt{m!}} \frac{z^{*m}}{\sqrt{m!}} |n\rangle\langle m| e^{-z^* z}$$

$$= \int_0^{\infty} \frac{\rho\, d\rho}{\pi} \int_0^{2\pi} d\theta \sum_{m,n} \frac{\rho^n e^{in\theta} \rho^m e^{-im\theta}}{\sqrt{n!}\sqrt{m!}}$$

$$e^{-\rho^2} |n\rangle\langle m|$$

$$= \int_0^\infty \frac{\rho\, d\rho}{\pi} \sum_{m,n=0}^{\infty} 2\pi\, \delta_{m,n} \frac{\rho^{n+m}}{\sqrt{n!\, m!}} e^{-\rho^2} |n\rangle\langle m|$$

$$= \sum_{n=0}^{\infty} 2 \int_0^\infty d\rho\, \rho\, \rho^{2n} \frac{1}{n!} e^{-\rho^2} |n\rangle\langle n|$$

$$(\rho^2 = u)$$

$$= \sum_{n=0}^{\infty} \int_0^\infty du\, u^n e^{-u} \frac{1}{n!} |n\rangle\langle n|$$

$$= \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{1}.$$

21.2.2

$$H = \sum_{n=1}^{N} \Big[E_0 |n\rangle\langle n| - t\left(|n\rangle\langle n+1| + |n+1\rangle\langle n| \right) \Big]$$

$$|\theta\rangle = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} e^{in\theta} |m\rangle$$

$$T|m\rangle = |n+1\rangle$$

$$T|\theta\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{in\theta} |n+1\rangle$$

$$= \frac{1}{\sqrt{N}} e^{-i\theta} \sum_{n=1}^{N} e^{i(n+1)\theta} |n+1\rangle$$

$$= e^{-i\theta} |\theta\rangle .$$

$$T = T \sum_{m=1}^{N} |n\rangle\langle n|$$

$$= \sum_{n=1}^{N} |n+1\rangle\langle n|$$

$$\langle n | T^\dagger = \langle n+1 |$$

$$T^\dagger = \sum_{n=1}^{N} |n\rangle\langle n| T^\dagger = \sum_{n=1}^{N} |n\rangle\langle n+1|$$

$$T^\dagger |n\rangle = \sum_{n'=1}^{N} |n'\rangle\langle n'+1|n\rangle = |n-1\rangle$$

$$H = E_0 - t\,(T + T^\dagger)$$

$$T^\dagger |\theta\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{in\theta} |n-1\rangle$$

$$= e^{+i\theta} |\theta\rangle$$

$$H|\theta\rangle = \left(E_0 - t\left(e^{-i\theta} + e^{i\theta}\right)\right) |\theta\rangle$$

$$= E(\theta)\,|\theta\rangle \quad \text{with}$$

$$E(\theta) = E_0 - 2t\cos\theta$$

The number of linearly independent states, i.e. the dimension of Hilbert space, is N.

We should require that replacing n by $n + N$ for all n leaves $|\theta\rangle$ unchanged

$$(T)^N |\theta\rangle = e^{iN\theta} |\theta\rangle = |\theta\rangle$$

$\therefore$ The allowed values of θ are then $\theta_k = k\frac{2\pi}{N}$ with $k = 0, 1, 2 \ldots N-1.$

For $N = 2$, there are two energy eigenstates

$\theta = 0$ $\quad |S\rangle = (|1\rangle + |2\rangle)\frac{1}{\sqrt{2}}$

with energy $E_S = E_0 - 2t$

and $\theta = \pi$ $\quad |A\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$

with energy $E_A = E_0 + 2t$

This is consistent with Eqs. (21.2.35) and (21.2.36) :