Lecture 2:

2.1 Many impurity system:

Consider a single electron scattering from many impurities randomly placed at $\vec{X}_i$:

$$H = \frac{p^2}{2m} + \sum_j U(x - \vec{X}_j)$$

$$= \sum_p \epsilon_p a_p^+ a_p + \sum_q U(q)\, \rho_q \sum_p a_{p+q}^+ a_p \quad , \quad \rho_q \equiv \sum_j e^{-i\vec{q}\cdot\vec{X}_j}$$

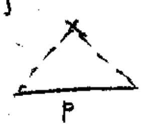
we will need to average over the impurity positions, giving

$$\bar{\rho}_q = \frac{N}{V}\int d^3x\, e^{-i\vec{q}\cdot x} = N\delta_{q,0}$$

$$\overline{\rho_{q_1}\rho_{q_2}} = \sum_{ij} \overline{e^{iq_1\cdot X_i}\, e^{-iq_2\cdot X_j}} = N^2\delta_{q_1,0}\,\delta_{q_2,0} + N\,\delta_{q_1+q_2,0} \qquad N \gg 1$$

In particular: $\overline{\rho_q \rho_{-q}} = N^2\delta_{q,0} + N$

Low density & weak scattering:

$\Sigma(\vec{p},\varepsilon) =$  $+$ 

1st order Born

$\rightarrow$ shift in energy: $\varepsilon_p \rightarrow \varepsilon_p + N.U(q=0) = \varepsilon_p + n_{imp}\int U(x)\, d^3x$

2nd order: $\quad \Sigma(p,\varepsilon) = N\sum_{p'} U^2(p'-p)\, G^0(p',\varepsilon)$

$$\operatorname{Im} G^0(p,\varepsilon) = -\pi \operatorname{sign}\varepsilon\; \delta(\varepsilon - \varepsilon_p)$$

$$\Rightarrow \Sigma = \Delta - i\pi \operatorname{sign}\varepsilon .\, N\sum_{p'} U^2(p'-p)\,\delta(\varepsilon - \varepsilon_{p'})$$

$$\sum_p \rightarrow N_0\int d\xi_p \int \frac{d\Omega_p}{4\pi} \quad \Rightarrow \quad \Sigma = \Delta - \frac{i\operatorname{sign}\varepsilon}{2\tau}$$

where $\quad \frac{1}{2\tau} = \pi N_0 n_{imp}\int \frac{d\Omega_{p'}}{4\pi} U^2(p'-p) \rightarrow \pi N_0 n_{imp} U^2$

Note: 2nd Born already introduces irreversibility

2.2 : Conductivity : Linear response

Define current correlation function

$$L_{\alpha\beta}(i\Omega_m) = \int_0^\beta d\tau \, e^{i\Omega_m \tau} \langle T_\tau [j_\alpha(\tau) j_\beta(\tau)] \rangle \quad , \quad \alpha, \beta = x, y$$

where $\Omega_m = 2\pi T m$ is the bosonic Matsubara frequency, $j(\tau)$ is the current density in Heisenberg representation, and $\langle \; \rangle$ denotes thermal average.

The conductivity is given by $\sigma_{\alpha\beta} = \lim \frac{1}{i\Omega_m} L_{\alpha\beta}(i\Omega_m)$, or

$$\sigma_{\alpha\beta}(\Omega) = i \frac{ne^2}{m\Omega} \delta_{\alpha\beta} + \frac{2e^2}{m^2\Omega} T \sum_p \sum_{p'} \sum_{\varepsilon_n} \langle G(p, p'; i\varepsilon_n + \tfrac{i\Omega_m}{2}) \, G(p', p; i\varepsilon_n - i\Omega/2) \rangle \, p_\alpha p'_\beta$$

where the 2-particle Green's function satisfies the equation

$$[\varepsilon - H_0 - V(r)] \, G_\varepsilon(\vec{r}, \vec{r}\,') = \delta(\vec{r} - \vec{r}\,')$$

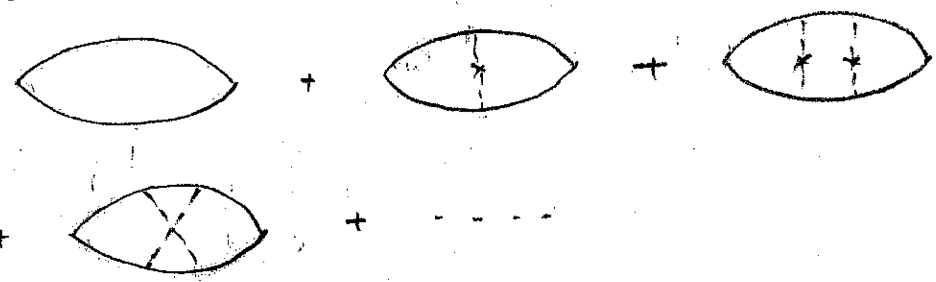
Averaged over the impurity distribution:

$$\langle G_\varepsilon(r, r') \rangle = \sum_p G(p, \varepsilon) \, e^{i\vec{p}\cdot(\vec{r} - \vec{r}\,')}$$

$$\text{where} \quad G(p, \varepsilon) = \frac{1}{\varepsilon - \xi_p + i \, \mathrm{sgn}\,\varepsilon / 2\tau} \quad , \quad \xi_p = \frac{p^2}{2m} - \mu$$

Note : $\langle GG \rangle \neq \langle G \rangle \langle G \rangle$. Using diagrammatic expansion,

$\langle GG \rangle =$



Leading contribution : $\langle GG \rangle = \langle G \rangle \langle G \rangle$

Case I : $\varepsilon + \Omega/2 > 0, \quad \varepsilon - \Omega/2 > 0, \quad \varepsilon > 0$

$$T \sum_{\varepsilon_n} \sum_{p} G_p^+ (i\varepsilon_n + i\Omega_m/2)\, G_p^+ (i\varepsilon_n - i\Omega_m/2) = \lim_{D\to\infty} N_0 \int_{-D}^{D} d\xi_k \int_0^\infty \frac{d\varepsilon}{2\pi} \frac{1}{\varepsilon + \Omega/2 - \xi_k + i/2\tau} \frac{1}{\varepsilon - \Omega/2 - \xi_k + i/2\tau}$$

Prob 2.1: Show that this cancels the singular term $\frac{ine^2}{m\Omega}$ in σ_{xx}, where $n = k_F^3/6\pi^2$

Case II : $\varepsilon + \Omega/2 > 0, \quad \varepsilon - \Omega/2 < 0$,

$$T \sum_{\varepsilon_n} \sum_{p} G_p^+ (i\varepsilon_n + i\Omega_m/2)\, G^- (i\varepsilon_n - i\Omega_m/2) =$$

$$\approx \int_{-\Omega/2}^{\Omega/2} \frac{d\varepsilon}{2\pi} \quad N_0 \int_{-\infty}^{\infty} d\xi_k \frac{1}{\varepsilon + \Omega/2 - \xi_k + i/2\tau} \frac{1}{\varepsilon - \Omega/2 - \xi_k - i/2\tau}$$

Prob 2.2: Show that this gives the known Drude contribution

$$\sigma = 2 \frac{ne^2\tau}{m} \frac{1}{1 - i\Omega\tau}$$

Note: Including $G_{\vec{p}} G_{\vec{p}'} P_\alpha P_\beta'$ gives rise to angular integrals which modifies $\tau \to \tau_{tr}$, where

$$\frac{1}{\tau_{tr}} = \frac{n m p_F}{(2\pi)^2} \int \underbrace{|V(\theta)|^2}_{\text{impurity potential}} (1 - \cos\theta)\, d\Omega$$

Soln 2.1

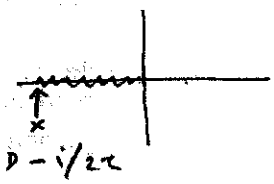
$$N_0 \int d\xi_k \int_0^\infty d\varepsilon \; \frac{1}{\varepsilon + \Omega/2 - \xi_k + i/2\tau} \; \frac{1}{\varepsilon - \Omega/2 - \xi_k + i/2\tau}$$

$$= N_0 \frac{1}{\Omega} \int_{-D}^{D} d\xi_k \; \ln \frac{\xi_k + \Omega/2 - i/2\tau}{\xi_k - \Omega/2 - i/2\tau} =$$

$$= (D + \Omega/2 - i/2\tau) \ln (D + \Omega/2 - i/2\tau) - (D - \Omega/2 - i/2\tau) \ln (D - \Omega/2 - i/2\tau)$$
$$- (-D + \Omega/2 - i/2\tau) \ln (-D + \Omega/2 - i/2\tau) - (-D - \Omega/2 - i/2\tau) \ln (-D - \Omega/2 - i/2\tau)$$

$$\simeq (D + \Omega/2 - i/2\tau) \left[\ln D + \frac{\Omega/2 - i/2\tau}{D} \right] - (D - \Omega/2 - i/2\tau) \left[\ln D - \frac{\Omega/2 + i/2\tau}{D} \right]$$
$$- (-D + \Omega/2 - i/2\tau) \left[\ln(-D) - \frac{\Omega/2 - i/(2\tau)}{D} \right] + (-D - \Omega/2 - i/2\tau) \left[\ln(-D) + \frac{\Omega/2 + i/(2\tau)}{D} \right]$$

$$= \Omega \ln D - \Omega \ln(-D) = \Omega \ln\left(\frac{D}{-D}\right) = \Omega(-i\pi)$$



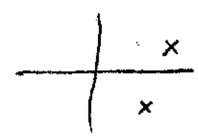
$$\sigma_{xx} : \quad \frac{\cancel{2}e^2}{m^2\Omega} \frac{N_0}{\cancel{2\pi}\cancel{\Omega}} (-i\cancel{\pi}\cancel{\Omega}) \frac{1}{3} p_F^2 \cdot = \frac{-ie^2}{\Omega} \frac{p_F^2}{3m\cancel{m}} \cdot \frac{\cancel{m}p_F}{2\pi^2} \qquad N_0 = \frac{mp_F}{2\pi^2}$$

$$= -\frac{ie^2}{m\Omega} \cdot n \qquad n = \frac{p_F^3}{6\pi^2}$$

$\rightarrow$ cancels the 1st term.

Soln 2.2

$$\int_{-\Omega/2}^{\Omega/2} \frac{d\varepsilon}{2\pi} \cdot N_0 \int_{-\infty}^{\infty} d\xi_k \; \frac{1}{\varepsilon + \Omega/2 - \xi_k + i/2\tau} \; \frac{1}{\varepsilon - \Omega/2 - \xi_k - i/2\tau}$$



$$= \int_{-\Omega/2}^{\Omega/2} \frac{d\varepsilon}{\cancel{2\pi}} \cdot N_0 \cdot \cancel{2\pi} i \cdot \frac{1}{+\frac{i}{\tau} + \Omega} = \Omega \cdot N_0 \cdot \frac{1}{\frac{1}{\tau} - i\Omega}$$

$$\sigma_{xx} : \quad \frac{2e^2}{m^2\cancel{\Omega}} \cdot \frac{\cancel{\Omega} N_0 \tau}{1 - i\Omega\tau} \cdot \frac{1}{3} p_F^2 = \underset{\substack{\uparrow \\ \text{spin}}}{2} \cdot \frac{ne^2\tau}{m} \frac{1}{1 - i\Omega\tau} \qquad \left(= 2 \cdot e^2 N_0 \cdot \frac{1}{3} v_F^2 \tau \Big/ (1 - i\Omega\tau) \right)$$