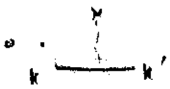
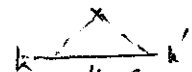
Lecture 5:

5.1 : Short range limit

We could start directly with $V(\vec{k}-\vec{k}') = V_0$

Then defining $(\pi N_0 V_0)^2 \equiv w$ and $(g/2)^2 w \equiv u$, we have

$\pi N_0 \cdot$  $= \pi N_0 V_0 - i V_0 \pi N_0\, g\, (\hat{k}\times\hat{k}')_z = \sqrt{w} - i\,2\sqrt{u}\,(\hat{k}\times\hat{k}')$

πN_0  $= \sum_{k_1} G_{k_1} \left[\sqrt{w} - i2\sqrt{u}\,(\hat{k}\times\hat{k}_1)_z\right]\left[\sqrt{w} - i2\sqrt{u}\,(\hat{k}_1\times\hat{k}')\right]$

$$= -i\,\mathrm{sign}\,\varepsilon_n \left[w - 4u \underbrace{\left\langle (\hat{k}\times\hat{k}_1)_z (\hat{k}_1\times\hat{k}')_z \right\rangle_{k_1}}_{\text{ave. over angles } = -\frac{1}{2}\hat{k}\cdot\hat{k}'} \right]$$

$$= -is\left[w + 2u\,\hat{k}\cdot\hat{k}'\right]$$

$\pi N_0 \cdot$  $= -is \sum_{k_2} \left[w + 2u\,\hat{k}\cdot\hat{k}_2\right]\left[\sqrt{w} - i2\sqrt{u}\,(\hat{k}_2\times\hat{k}')\right] \cdot G_{k_2}$

$$= (-is)^2 \left[w\sqrt{w} - i4u\sqrt{u} \underbrace{\left\langle (\hat{k}\cdot\hat{k}_2)(\hat{k}_2\times\hat{k}') \right\rangle_{k_2}}_{\frac{1}{2}(\hat{k}\times\hat{k}')_z} \right]$$

$$= (-i)^2 \left[w\sqrt{w} - i2u\sqrt{u}\,(\hat{k}\times\hat{k}')\right]$$

$\pi N_0 \cdot$  $= (-i)^3 s \left[w^2 + 2u^2(\hat{k}\cdot\hat{k}')\right]$

$+ \cdots$

$$\text{Sum } \pi N_0 f_{kk'} = \frac{\sqrt{w}}{1+w} - i\,\frac{2\sqrt{u}}{1+u}(\hat{k}\times\hat{k}') - is\left[\frac{w}{1+w} + \frac{2u}{1+u}(\hat{k}\cdot\hat{k}')\right]$$

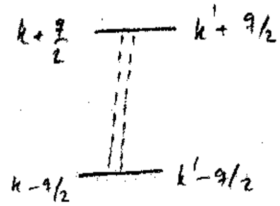
Use $(\hat{k}\times\hat{k}')_z = \frac{1}{2i}(k_- k_+' - k_+ k_-')$ $\qquad k_\pm = k_x \pm i k_y = e^{\pm i\varphi}$

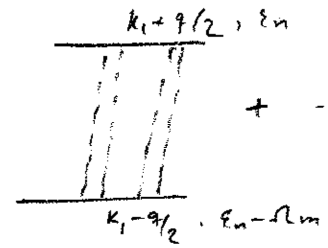
$(\hat{k}\cdot\hat{k}') = \frac{1}{2}(k_- k_+' + k_+ k_-')$

$$\Rightarrow \bar{f}_{kk'} = \underbrace{\left(\frac{\tilde{w}}{\sqrt{w}} - is\tilde{w}\right)}_{\bar{f}_0} + \underbrace{\left(-is\tilde{u} + \frac{\tilde{u}}{\sqrt{u}}\right)}_{\bar{f}_1}\hat{k}_+\hat{k}_-' + \underbrace{\left(-is\tilde{u} - \frac{\tilde{u}}{\sqrt{u}}\right)}_{\bar{f}_{-1}}\hat{k}_-\hat{k}_+'$$

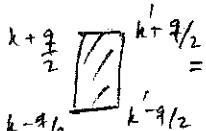
Prob 5.1: Show that this agrees with $\bar{f}^s_{m\sigma}$ defined in Prob. 4.2

5.2: Particle-hole propagator:

$\Gamma_{kk'}(q; i\varepsilon_n, i\Omega_m) =$ 

$+$ 

$+ \cdots$



$= t_{kk'} + \sum_{k_1} t_{kk_1} G_{k_1+q/2,\sigma}(i\varepsilon_n)\, G_{k_1-q/2,\sigma}(i\varepsilon_n - i\Omega_m)\, \Gamma_{k_1 k'}$

Define $\bar{\Gamma}_{kk'} \equiv 2\pi N_0 \tau_0 \Gamma_{kk'} = \sum_m \bar{\Gamma}_{mm'} \chi_m(\hat{k}) \chi_{m'}(\hat{k}')$

For $s' = -s$ and defining $\bar{t}^{+-}_{m\sigma} \equiv \lambda_m$, we get

$$\bar{\Gamma}^{ss'}_{mm'} = \lambda_m \delta_{mm'} + \lambda_m \Big\{ \big[1 - \tau\big(|\Omega_m| + D_0 q^2\big) \big] \bar{\Gamma}^{ss'}_{mm'}$$
$$- \frac{i}{2} v_F q \tau s \big[\bar{\Gamma}^{ss'}_{m-1,m'} \chi_1^*(\hat{q}) + \bar{\Gamma}^{ss'}_{m+1,m'} \chi_1(\hat{q}) \big]$$
$$- \frac{1}{4}(v_F q \tau)^2 \big[\bar{\Gamma}^{ss'}_{m-2,m'} \chi_2^*(\hat{q}) + \bar{\Gamma}^{ss'}_{m+2,m'} \chi_2(\hat{q}) \big] \Big\}$$

Here we need $\int d\varepsilon_1\, G^{+}_{k_1+q/2,\sigma}(i\varepsilon_n)\, G^{-}_{k_1-q/2,\sigma}(i\varepsilon_n - i\Omega_m)$, $\varepsilon_n > 0$, $\varepsilon_n - \Omega_m < 0$

$$= 2\pi\tau \big[1 + i\tau(i\Omega_m - \vec{q}\cdot\vec{v}_{k_1}) - \tau^2 (\vec{q}\cdot\vec{v}_{k_1})^2 \big] + O(q^3)$$

with $\vec{q}\cdot\vec{v}_{k_1} = q v_F (\hat{q}\cdot\hat{k}_1)$

Note: For $m = m' \neq 0$: $\bar{\Gamma}_{mm} = \dfrac{\lambda_m}{1-\lambda_m}$

The case $m = 0$ needs special care. In general, in the regime $v_F q \tau < 1$, one obtains

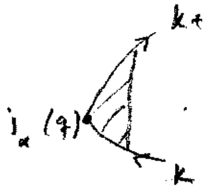
$$\boxed{\bar{\Gamma}_{kk'} = \frac{1}{\tau} \frac{\gamma_k \tilde{\gamma}_{k'}}{|\Omega_m| + D q^2} + \sum_{m \neq 0} \tilde{\lambda}_m \chi_m(\hat{k}) \chi_m^*(\hat{k}')}$$

where $\gamma_k \equiv 1 - \frac{i}{2} v_F \tau s \sum_{m=\pm 1} \tilde{\lambda}_m \chi_m(\hat{k})\, q_{-m}$ | $q_\pm \equiv q_x \pm i q_y$

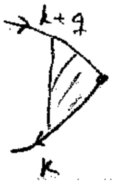
$\tilde{\gamma}_k \equiv 1 - \frac{i}{2} v_F \tau s \sum_{m=\pm 1} \tilde{\lambda}_m \chi_m^*(\hat{k})\, q_m$ | $\tilde{\lambda}_m \equiv \dfrac{\lambda_m}{1-\lambda_m}$

$D \equiv D_0 (1 + \mathrm{Re}\,\tilde{\lambda}_1)$, $D_0 = \frac{1}{2} v_F^2 \tau$

5.3: Renormalized current vertices



$$j_\alpha(q) = v_{k\alpha} + \langle v_{k'\alpha} \bar{\Gamma}_{k'k} \rangle_{k'}$$
$$= v_{k\alpha} + \langle v_{k'\alpha} \gamma_{k'} \rangle_{k'} \frac{1/\tau}{|\Omega_m + Dq^2} \tilde{\gamma}_k + \sum_{m=\pm 1} \tilde{\lambda}_m \chi_m^*(\hat{k}) \langle v_{k'\alpha} \chi_m(\hat{k}') \rangle_{k'}$$



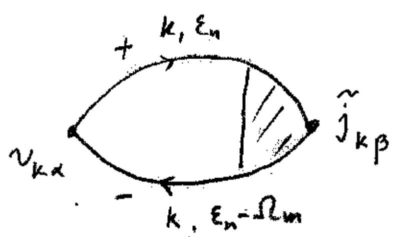
$$\tilde{j}_\alpha(q) = v_{k\alpha} + \langle v_{k'\alpha} \bar{\Gamma}_{kk'} \rangle_{k'}$$
$$= v_{k\alpha} + \langle v_{k'\alpha} \tilde{\gamma}_{k'} \rangle_{k'} \frac{1/\tau}{|\Omega_m + Dq^2} \gamma_k + \sum_{m=\pm 1} \tilde{\lambda}_m \chi_m(\hat{k}) \langle v_{k'\alpha} \chi_m^*(\hat{k}') \rangle_{k'}$$

Prob 5.2: Show that $j_{kx}(q=0) = v_F \left[(1 + \mathrm{Re}\,\tilde{\lambda}_1) \hat{k}_x - \mathrm{Im}\,\tilde{\lambda}_1 \hat{k}_y \right]$.

Note: In general:

$$j_{k\alpha} = v_F \left[(1 + \mathrm{Re}\,\tilde{\lambda}_1) \hat{k}_\alpha + \mathrm{Im}\,\tilde{\lambda}_1 (\hat{e}_\alpha \times \hat{k})_z \right]$$
$$\tilde{j}_{k\alpha} = v_F \left[(1 + \mathrm{Re}\,\tilde{\lambda}_1) \hat{k}_\alpha - \mathrm{Im}\,\tilde{\lambda}_1 (\hat{e}_\alpha \times \hat{k})_z \right]$$

5.4: Conductivity: Skew scattering



$$L_{\alpha\beta} = T \sum_{\varepsilon_n} \sum_{k,\sigma} G_{k\sigma}(i\varepsilon_n) G_{k\sigma}(i\varepsilon_n - i\Omega_m) v_{k\alpha} \tilde{j}_{k\beta}$$

$\Omega_m > 0$

Prob 5.3: Show that the conductivity tensor is given by

$$\sigma_{\alpha\beta}^{ss} = \sum_\sigma \frac{1}{2} v_F^2 \tau_\sigma N_\sigma \begin{pmatrix} 1 + \mathrm{Re}\,\tilde{\lambda}_1 & \mathrm{Im}\,\tilde{\lambda}_1 \\ -\mathrm{Im}\,\tilde{\lambda}_1 & 1 + \mathrm{Re}\,\tilde{\lambda}_1 \end{pmatrix}$$

"Anomalous Hall" no $\vec{B}$-field

Hint: the energy integral over GG is nonzero only if the poles are on opposite sides of real axis, requiring $0 \le \varepsilon_n \le \Omega_m$ for $\Omega_m > 0$ and gives $2\pi N_\sigma \tau_\sigma$, while the sum over ε_n gives $\Omega_m / 2\pi T$.