PHZ 3113, Section 3924, Fall 2013, Homework Bonus

Due at the start of class on Friday, September 27. Half credit will be available for homework submitted after the deadline but no later than the start of class on Monday, September 30.

Answer all questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. Consider:
$$\phi(x, y, z) = \frac{(x^2 + z^2)}{2(x^2 + y^2 + z^2)^2}.$$
   a) Evaluate the Cartesian components of $\nabla\phi$.
   b) Find $|\nabla\phi|$, and $\mathbf{r} \cdot \nabla\phi$.
   c) Evaluate $\nabla^2\phi$.

2. This question concerns Green's theorem.
   a) For a circle of raduis R centered at the origin, use Green's theorem to show that:
$$\frac{1}{2}\oint_C (xdy - ydx) = A,$$
   where C is the circumference of the circle, and A is its area.
   b) By using plane polar coordinates, and the appropriate change of variables for each part of the integral in part a), show that you obtain the expected area for the circle.

3. This question concerns a simple Green's function.
   a) Use the variation of parameters method to show that
$$y(t) = \frac{1}{\omega}\int_0^t \sin(\omega(t - t')f(t')dt',$$
   solves the following differential equation with $y(0) = 0$ and $\dot{y}(0) = 0$:
$$\ddot{y}(t) + \omega^2 y(t) = f(t). \qquad (*)$$
   b) By carefully considering the differentiation at $t = t'$, and using the relation between the Heaviside step function, $\theta(t - t')$, and the Dirac delta, $\delta(t - t')$, show that:
$$G(t, t') = \begin{cases} 0 & 0 < t < t', \\ \frac{1}{\omega}\sin(\omega(t - t')) & 0 < t' < t, \end{cases}$$
   meets the requirements for being a Green's function for $(*)$ in that it exactly satisfies:
$$\ddot{G}(t, t') + \omega^2 G(t, t') = \delta(t - t'). \qquad (**)$$
   c) Show that $\tilde{G}(t, t') = G(t, t') + 1/(2\omega)\sin(\omega(t' - t))$ also satisfies $(**)$. Note: however, $\tilde{y}(t) = \int_0^\infty \tilde{G}(t, t')f(t')dt'$ would generally have different behavior at $t = 0$ from $y(t)$. What property of the added term allows $\tilde{G}(t, t')$ to also be a Green's function for $(*)$?