

## PHZ 3113, Section 3924, Fall 2013, Homework 5

**Due at the start of class on Friday, October 4.** Half credit will be available for homework submitted after the deadline but no later than the start of class on Monday, October 7.

*Answer all questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.*

1. Consider the differential equation:

$$y''(x) - \frac{1}{(x-1)}y'(x) + \frac{1}{x(x-1)}y(x) = f(x).$$

- a) Show that  $y_1(x) = x$  is a solution when  $f(x) = 0$ .
  - b) Find a linearly independent solution  $y_2(x)$  such that  $y_2(0) = 1$  when  $f(x) = 0$ .
  - c) Find the solution with  $y(0) = 0$  and  $y'(0) = 1$  when  $f(x) = 2(1-x)$ .
2. This question concerns parabolic cylinder coordinates, which are related to Cartesian coordinates by:
- $$\begin{aligned} x &= \frac{1}{2}(u^2 - v^2), \\ y &= uv. \end{aligned}$$
- a) Find the Jacobian  $J$  of the transformation to the  $(u, v)$  coordinates.
  - b) Substitute for  $dx$  and  $dy$  in  $d\mathbf{s} = \mathbf{e}_x dx + \mathbf{e}_y dy$  and find unit vectors pointing in the  $\mathbf{e}_u$  and  $\mathbf{e}_v$  directions. Hence, identify  $h_u$  and  $h_v$  in  $d\mathbf{s} = h_u \mathbf{e}_u du + h_v \mathbf{e}_v dv$ .
  - c) Show that  $\mathbf{e}_u$  and  $\mathbf{e}_v$  are orthogonal.
  - d) By evaluating  $ds^2 = d\mathbf{s} \cdot d\mathbf{s}$  find  $g_{uu}$  and  $g_{vv}$  and show that  $g_{uv} = 0$ .
  - e) Calculate the determinant of  $g_{ij}$  and relate it to  $J$ . This relation is universal.
3. Let  $\Gamma$  be the circulation of the vector field with cartesian components  $(x^2y, xy^2, z)$  around the perimeter of the ellipse  $x^2/a^2 + y^2/b^2 = 1$ ,  $z = 0$ , traveling counterclockwise when viewed from positive values of  $z$ .
- a) Calculate  $\Gamma$  by direct evaluation of the circulation.
  - b) Calculate  $\Gamma$  by using Stokes' Theorem.

Hint: Change variables using  $x = ar \cos(\phi)$ ,  $y = br \sin(\phi)$ .

4. Let  $\Phi$  be the outward flux of the vector field  $\mathbf{F} = (y-z)\hat{\mathbf{x}} + x^2y\hat{\mathbf{y}}$  through the region  $z > 0$  of the surface of a sphere of radius  $R$  centered on the origin.
- a) Calculate  $\Phi$  by direct evaluation of the flux.
  - b) Calculate  $\Phi$  by converting the open surface to a closed one, and then applying the divergence theorem.