PHZ 3113, Section 3924, Fall 2013, Homework 9

Due at the start of class on Wednesday, November 6. Half credit will be available for homework submitted after the deadline but no later than the start of class on Wednesday, November 13.

Answer all questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. The equation $|\mathrm{M} - \lambda \mathrm{I}| = 0$ for the eigenvalues of a matrix, M, is called the *characteristic equation.* Every matrix satisfies its own characteristic equation. Show explicitly that this is true for:
$$\mathrm{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
i.e., that:
$$\mathrm{M}^2 - Tr(\mathrm{M})\mathrm{M} + Det(\mathrm{M})\mathrm{I} = 0.$$

2. Consider an arbitrary, 2×2 matrix with complex coefficients:
$$\mathrm{M} = \begin{pmatrix} a + ib & c + id \\ e + if & g + ih \end{pmatrix}.$$
Find the condition that M be unitary, *i.e.,* that $\mathrm{M}^\dagger = \mathrm{M}^{-1}$. Can you characterize M exclusively in terms of a number of angles?

3. Let $\mathrm{V}^T = (x, y)$, and let M be a real, symmetric 2×2 matrix. Then $\mathrm{V}^T \mathrm{M} \mathrm{V} = K$, represents a central conic section (ellipse or hyperbola). Find (x', y'), a rotation of the original coodinates, and $\{\lambda_1, \lambda_2 \geq \lambda_1\}$, such that the equation for the conic section becomes $\lambda_1 x'^2 + \lambda_2 y'^2 = K$, if
$$\mathrm{M} = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}.$$

4. Let a mass of $4m$ hang from a suspension point by a spring of spring constant $3k$. Let a mass of m hang from the first mass with a spring of spring constant k. Find the eigenvalues and eigenvectors for the problem and solve for the equations of motion.
(Hint: Let x and y be the displacements of the two masses from *equilibrium*, in which case the effects of gravity can be ignored.)