Methods of Modern Numerical Relativity

Benjamin Hall
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General Relativity Class Presentation
Outline

- Historical Overview
- Areas of Concern
  - Formulations (ADM, BSSN, Conformal, etc)
  - Constraints
  - Gauge Conditions
    - Gauge Shocks
  - Boundary Conditions/Excision
- Recent Results
Timeline for Numerical Relativity

- 1916: GR introduced
- 1936: ADM formulation
- 1956:
- 1966:
- 1976: "Standard" ADM
- 1986:
- 1996: BSSN
- 2000: Modern Era

Early Years:
- ~35 hits

2008:

Modern Era:
- ~1500 hits
What makes a good formulation?

- Stability
  - Hyperbolicity and well-posedness
- Ease of implementation
- Physical Meaning
Well-posedness of a system of differential equations guarantees:
- existence of (local) solutions
- uniqueness of solutions
- continuous dependence of solutions on initial data.

Ill-posed systems may have solutions, but tend towards instability independent of numerical algorithms.
Hyperbolic equations

- Characteristic Matrix
  \[ M^i_j = \partial_j f^i \]
- If eigenvalues of $M > 0$: Weakly Hyperbolic
  - Not generally well-posed
- Eigenvalues of $M > 0$ and $M$ diagonalizable: Strongly Hyperbolic
  - Can be well-posed for many types.
- $M$ Hermitian: Symmetric Hyperbolic
  - Almost always well-posed
Arnowitt-Deser-Misner (ADM)[1]

- $ds^2 = -\alpha^2 dt^2 + \gamma^{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt)$
- Foliates space-time into a series of spacelike slices reaching to spatial infinity.
- Dynamical variables ($\gamma^{ij}$, $K^{ij}$)
- Used by most groups from 1977 till 1999; superseded by the BSSN formulation
- Weakly hyperbolic and ill-posed
- Unstable even with good gauge and boundary conditions.

Baumgarte-Shapiro-Shibata-Nakamura (BSSN)[1]

- Strongly hyperbolic reformulation of ADM
- Dynamical Variables \((\phi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \Gamma^i)\)
- Significantly more stable

\[
\phi = \frac{1}{12} \log(\det \gamma_{ij})
\]
\[
\tilde{\gamma}_{ij} = e^{-4\phi} \gamma_{ij}
\]
\[
K = \gamma_{ij} K_{ij}
\]
\[
\tilde{A}_{ij} = e^{-4\phi} \left( K_{ij} - \frac{1}{3} \gamma_{ij} K \right)
\]
\[
\tilde{\Gamma}^i = \tilde{\Gamma}^i_{jk} \gamma^{jk}
\]

Other Formulations

- **Conformal**
  - Uses slices reaching to future null infinity
  - Allows compactification and more natural boundary conditions

- **Characteristic**
  - Foliates using characteristic cones from a central world-tube
  - Also allows compactification and natural boundary conditions
Conformal Diagram for Formulations

Conformal Slices
Constraints

- Equations with no time derivatives are called constraints.
- Analytically, if the initial data obeys the constraints, all future evolution will still obey.
  - Not true when numerically evolved.
- Constraints must be met to keep solutions physical and stable.
Methods for constraints

- Fully constrained evolution
  - very expensive but stable
- Free evolution
  - cheap, but only works short-term
- Constraint-dampening evolution
  - mixed results, often more stable than free evolution
Importance of Constraints

Image from Shinkai, arXiv:0805.0068v1 [gr-qc]
Gauge Conditions

- Bona-Masso slicing
- 1+log slicing
- Maximal slicing
- Trivial shift
- Co-rotating frames
- Minimal distortion
Gauge Shocks

• Coordinate pathology
  • Characterized by discontinuities in lapse function and shift vector.
  • Caused by finite gauge propagation speeds or bad gauge choices

Taken from M. Alcubierre, arXiv:0503030v2 [gr-qc]
Boundary Conditions (3+1)

- ADM/BSSN vs Conformal/Characteristic
- Placed at edge of computational region
- “Constraint preserving” conditions
- “No inflow” conditions
- “Maximal Damping” conditions
Singularity Excision

- Removes regions from grid.
- Usually cut at apparent horizons.
- Involves interior boundaries.
- Moving singularities.
Recent Results—One Example

• Radiation Kicks (as discussed by Mitryk)