The Holographic Principal and its Interplay with Cosmology

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*Final Presentation: General Relativity*

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for simplicity, use the Schwarzschild metric

\[ ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\Omega \]

take a scenario with a stationary observer just outside of the horizon such that

\[ r = 2M + \frac{u^2}{2M} \]

rewrite the metric

\[ ds^2 = -\frac{u^2}{4M^2} dt^2 + 4du^2 + dX^2_{\perp} \]

\[ \tau = \frac{t}{4M} \quad \rho = 2u \quad ds^2 = -\rho^2 d\tau^2 + d\rho^2 + dX^2_{\perp} \]

d this configuration is known as \textit{Rindler Coordinates}
**Unruh effect**
An observer moving with uniform acceleration through the Minkowski vacuum observes a thermal spectrum of particles

\[ \beta(u) = 2\pi \rho = 4\pi u = 4\pi \sqrt{2M(r - 2M)} \]

this is subsequently red-shifted

\[ \beta(r') = 4\pi \sqrt{2M(r - 2M)} \sqrt{\frac{1 - \frac{2M}{r'}}{1 - \frac{2M}{r}}} \]

take the limit as \( \beta(r' \to \infty) = 4\pi \sqrt{2Mr} \quad R = 2M \)

at the horizon’s edge \( \beta(r' \to \infty) = 8\pi M \)

recalling thermodynamics \( \beta(r' \to \infty) = \frac{1}{kT} = 8\pi M \)

finally, the temperature

\[ T = \frac{1}{8\pi k_b M} \]

\[ T = \frac{\hbar c^3}{8\pi GMk_b} \text{ human units} \]
What is the Entropy?

\[ dS = \frac{dQ}{T} = 8\pi M dQ \quad \text{letting } k_b = 1 \]

exploiting the equivalence principal, take the energy added as a change in mass \( dM \)

\[ dS = 8\pi M dM = 8\pi d(M^2) \quad R = 2M \]

\[ S = \pi R^2 = \frac{A}{4} \]

- the entropy of the black hole is proportional to the area of the horizon
Introduction

- holographic principle is a contemporary question in physics literature (see references)
- holographic principal provides complications when being connected to cosmological models
- will explore some of the successes, failures, and “paradoxes” of the holographic principle

Example with GUT scale inflation:

\[ T \sim 10^{-3} \quad L_H \sim 10^{36} \quad A \sim L_H^2 \sim 10^{72} \quad S \sim T^3 L_H^3 \sim 10^{99} \]

Not consistent with Holographic Principle
The Holographic Principle

Seemingly Innocuous Statment: The description of a spatial volume of a black hole is encoded on the boundary of the volume.

Extreme Extension: The universe could be seen as a 2-D image projected onto a cosmological horizon.
Expanding Flat Universe

- common solution to any intro to general relativity text -- including our own
- provides simple test for the holographic principal

Start with the metric for a flat, adiabatically expanding universe:

\[ ds^2 = -dt^2 + a^2(t)dx^i dx^i \]

\[ \frac{\dot{\rho}}{\rho} + 3H(1 + \gamma) = 0 \]

where \( H = \frac{\dot{a}}{a} \)

\[ p = \gamma \rho \]
Radiation Dominated (FRW) Universe

Solutions to this are known... some highlights:

\[ a(t) = t^{\frac{3}{3(\gamma+1)}} \quad \text{let } a(1) = 1 \quad \rho = \frac{4}{2(1+\gamma)^2} \frac{1}{a^{3(1+\gamma)}} \]

To get the size of the universe, integrate a null geodesic to the boundary (which can be chosen as radial)

\[ \chi_H(t) = a(t) \int_0^t \frac{dt'}{a(t')} = a(t)r_H(t) \quad \text{such that } r_H(t) = \frac{dt}{a(t)} \]

So what’s the answer? These solutions will vary depending on the value of gamma, so let’s take \( \gamma > -\frac{1}{3} \).

\[ \chi_H = a(t)r_H(t) = \frac{3(\gamma + 1)}{3\gamma + 1} t \]

- integral starts at 0 -- looks like we cheated
  
  We get around the singularity at 0 by starting the integral at \( t=1 \). This corresponds to Planck time.
Don’t Forget about the entropy

At the start time, we have \( \chi_H(1) \sim 1 \)

This is at least getting us somewhere. We can use this to limit the entropy density since the volume of space will also be \( \sim 1 \)

\[
\frac{S}{A} = \sigma \sim 1
\]

we know the area of the horizon \( \sim \chi_H^2 \)

and the entropy of the horizon \( \sim \sigma r_H^2 \)

Recall this is an adiabatic expansion...

\[
\frac{S}{A}(t) \sim \sigma \frac{r_H^3}{\chi_H^2} = \sigma t^{\frac{\gamma-1}{\gamma+1}}
\]

Holographic Principle is upheld
What does this mean?

- found a case where the holographic principal holds...
- solution holds for $-1 < \gamma < 1$
Closed Universe

- closed universes tend to collapse...
- worth verifying with the Holographic Principle...

\[ ds^2 = -dt^2 + a^2(t)(d\chi^2 + \sin^2 \chi d\Omega) \]

\[ \frac{S}{A} = \sigma \frac{2\chi_H - \sin 2\chi_H}{2a^2(\chi_H) \sin^2 \chi_H} \]

- consider the case where the universe is dominated by dark matter: \( p \ll \rho \quad a = a_{\text{max}} \sin^2 \left( \frac{\chi_H}{2} \right) \)

- area vanishes at \( \chi_H = \pi \)

- \( \chi_H = \pi \) is regular...

**Holographic Principal is violated**
What Does All This Math Tell Us?

- topologically closed universes could not exist
- collapsing universes cannot happen in a manner that preserves the second law of thermodynamics
- entropy is always increasing
- if the universe is collapsing, the surface area must eventually vanish
So what *about* Thermodynamics?

- in the advent that the holographic principal cannot be consistent with the second law of thermodynamics, there are two choices
  - holographic principal is wrong?
  - thermodynamics is wrong?
- propose “Generalized Second Law of Thermodynamics”
  \[ S' = S_M + S_{BH} \]
  - effectively adding a correction term to work around writing off the entropy inside a black hole as unmeasurable
  - can no longer access the quantum states inside
Cosmological Inflation

- holographic principal and inflation are compatible depending on the rate of inflation
- could be that holographic principal bounds inflation or vice versa
- currently holographic bounds put $S \leq 10^{120}$ compared to current observations that give $S \leq 10^{90}$. 
• Let's take the situation where we have the universe extending into Anti de Sitter Space with the flat universe from earlier.

\[ \rho = \frac{\rho_0}{a^{3(\gamma+1)}} - \lambda \]

rewriting the Friedman equation yields

\[ \dot{a} = \pm \sqrt{\frac{\rho_0}{a^{3(\gamma+1)}} - \lambda a^2} \]

this has turning points when

\[ \rho_0 = \lambda a^{3(\gamma+1)} \]

solving for the turning points yields

\[ L_H(\dot{a} = 0) = \frac{B \left( \frac{\gamma}{2(\gamma+1)}, \frac{1}{2} \right)}{3(\gamma + 1)\sqrt{\lambda}} \]
leaving... \( L_H \sim \lambda^{-(3\gamma+1)/6(\gamma+1)} \gg 1 \)

- holographic principle becomes violated
- violation for holographic principal happens well before planck limit
- most of the success of the holographic principle comes from Anti de Sitter space / Conformal Field Theory (AdS/CFT) Correspondence
Conclusions

• holographic principle discovered in uncovering the temperature of a black hole

• has found a home in string theory and quantum gravity research (AdS/CFT correspondence)

• seems to have more of the characteristics of an anomaly than cause for quantum gravity

• does not work cosmologically for topologically closed universe / flat universe going to Anti de Sitter Space
References

2. Edward Witten, Anti De Sitter Space And Holography, hep-th/9802150
6. G’t Hooft, Dimensional Reduction in Quantum Gravity, gr-qc/9310006
7. Carroll, Spacetime and Geometry, 2004 Pearson
9. Ross, Black Hole Thermodynamics, hep-th/0502195
Backup Slides
Phase space of solutions

\[ \frac{\gamma-1}{\gamma+1} \]

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Laws of Black Hole Thermodynamics

- Stationary black holes have constant surface gravity.

1. \[ dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ \]
2. The horizon area is increasing with time
3. Black holes with vanishing surface gravity cannot exist.

http://en.wikipedia.org/wiki/Black_hole_thermodynamics