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(PHZ 6607 by Prof. B. Whiting)
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Vectors

In a manifold, M, a vector is an object associated with a single point in the manifold. To be precise, a vector at a point P in the manifold is an element of the tangent space at P. Now we will illustrate this definition. Let M be the

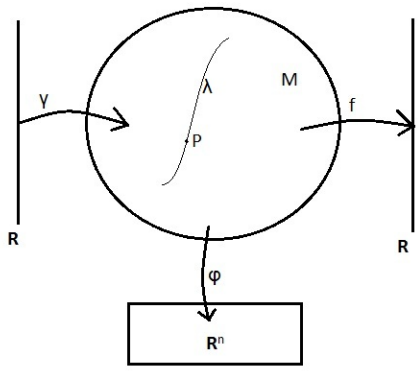

Figure 1: γ maps a curve in M which is parametrized by λ. f is a smooth function in M. ϕ maps from M into $\mathbf{R}^n$.

manifold. We want to construct the tangent space at a point P in the manifold. Now we consider the map $\gamma : \mathbf{R} \to M$ such that P is in the image of γ . Physically, γ describes a curve through P in M. Let λ be the parameter along the curve. Lastly, we define a smooth function f on M i.e. $f : M \to \mathbf{R}$. (Figure 1) The curve γ parametrized by λ defines a *directional derivative* of f at P, $(\frac{df}{d\lambda})_P$. So each curve through P corresponds to a directional derivative of f at P. Now we are in a position to formally define the tangent space at P, T_P :

T_P can be identified with the space of directional derivative operators along the curves through P

If the manifold M is n-dimensional, so will be T_P. Then we will have n independent directional derivatives of f at P.
The conventional basis of T_P is co-ordinate basis. For an n-dimensional manifold M, we consider n smooth functions on M, x^μ $(\mu = 1, 2, ..., n)$, so that $(\frac{dx^\mu}{d\lambda})_P$ are independent. x^μ are the *co-ordinates* on the manifold and $(\frac{dx^\mu}{d\lambda})_P$

are the elements of tangent space T_P. An element of $\mathbf{R}^n$ is specified by $(\frac{dx^\mu}{d\lambda})_P$ with $\mu = 1, 2, ..., n$.
Therefore, in a co-ordinate independent definition of vectors, vectors are differential operators.

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu \tag{1}$$

where $\frac{d}{d\lambda}$ is the abstract vector, $\frac{dx^\mu}{d\lambda}$ are the components of the vector and ∂_μ are the basis elements known as co-ordinate basis for the tangent space.
A vector acting on a one-form produces a scalar, while a vector acting on a scalar produces a vector.

$$V_\lambda(f) = \frac{df}{d\lambda} \quad : a \ \ vector \tag{2}$$

where f is a scalar and λ parametrizes a curve in the manifold.

Commutators of two vectors or Lie brackets

We defined a vector at some point P in a manifold M to be an element of the tangent space at that point (T_P). We generalize this definition into a definition for vector fields. A *vector field* defines a map from smooth functions to smooth functions all over the manifold.
Two vectors, $\mathbf{V}$ and $\mathbf{U}$, are defined as:

$$\mathbf{V}(f) = \frac{df}{d\lambda} \tag{3}$$

$$\mathbf{U}(f) = \frac{df}{d\eta} \tag{4}$$

where λ and η parametrize two curves through some point P in the manifold. The *commutator* or *Lie bracket* of these two vectors is defined as:

$$[\mathbf{U},\mathbf{V}](f) = \mathbf{U}(\mathbf{V}(f)) - \mathbf{V}(\mathbf{U}(f)) \tag{5}$$

since both $\mathbf{V}$ and $\mathbf{U}$ are elements of T_P, $[\mathbf{U},\mathbf{V}](f)$ is also an element of T_P. Therefore, there exists a vector $\mathbf{W}$ in T_P such that

$$\mathbf{W}(f) = [\mathbf{U},\mathbf{V}](f) \tag{6}$$

In co-ordinate basis:

$$\mathbf{W} = W^\mu \partial_\mu \tag{7}$$

$$W^\mu = U^\lambda \partial_\lambda V^\mu - V^\lambda \partial_\lambda U^\mu \tag{8}$$

Co-vectors or One-forms

Vectors are elements of tangent space(T_P). We define *co-tangent space* (T_P^*) as the set of linear maps $\omega : T_P \to \mathbf{R}$. A *co-vector* or *one-form* is an element of the co-tangent space. A vector field is defined by $\frac{d}{d\lambda}$ and a co-vector field is

defined by d. As a physical example, the gradient of a scalar f is a one-form. In co-ordinate basis:

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu \tag{9}$$

where df is the co-vector, $\frac{\partial f}{\partial x^\mu}$ are the components of the co-vector and dx^μ are the basis elements in the co-tangent space. Co-vectors act like a differential operator.

$$d = \frac{\partial}{\partial x^\mu} . dx^\mu \tag{10}$$

vector acting on a co-vector or one-form produces a scalar.

$$\frac{d}{d\lambda} . df = \frac{df}{d\lambda} \quad : a \;\; scalar \tag{11}$$

Dot product of two basis elements is:

$$dx^\mu . \frac{\partial}{\partial x^\nu} = \delta^\mu_\nu \tag{12}$$

where dx^μ are the basis elements of the co-tangent space(T^*_P) and $\frac{\partial}{\partial x^\nu}$ are the basis elements of the tangent space (T_P).

Tensors

A (k, l) *tensor* is a multilinear map from a collection of k co-vectors and l vectors to $\mathbf{R}$. We write the components and the basis elements of the tensor together as:

$$\mathbf{T} = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \ldots \nu_l} \partial_{\mu_1} \otimes ... \otimes \partial_{\mu_k} \otimes dx^{\nu_1} \otimes ... \otimes dx^{\nu_l} \tag{13}$$

The components of the tensor are obtained by acting the tensor on basis one-forms and vectors. Therefore

$$T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \ldots \nu_l} = \mathbf{T}(dx^{\mu_1}, ..., dx^{\mu_k}, \partial_{\nu_1}, ..., \partial_{\nu_l}) \tag{14}$$