Lecture on Sept 23, 2014
(PHZ 6607 by Prof. B. Whiting)
Sankha Subhra Chakrabarty
Luyi Yan

The Metric

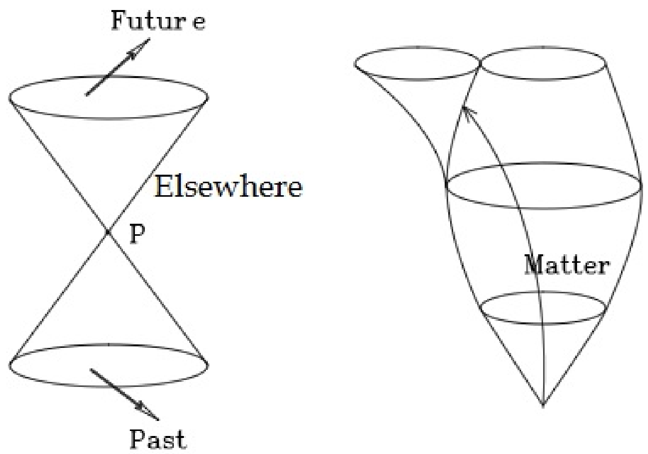
The *metric* is a (0,2) tensor with basis elements $dx^\mu \otimes dx^\nu$.

$$ds^2 = g = g_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

Here ds^2 is not the square of any differential, it is the name of the tensor called *metric tensor* whose components are $g_{\mu\nu}$. dx^μ also stands for the infinitesimal length along the basis vector. g also denotes the determinant of the matrix formed by $g_{\mu\nu}$. The space-time metric is not positive definite.

Let us provide a list of various uses of the metric.

1. The metric gives a local definition of future and past.



2. The metric allows us to compute the path length and proper time.
3. The metric determines the geodesic which is the extremal distance between two points in space-time.
4. The metric replaces the Newtonian gravitational field ϕ.
5. The metric provides a notion of locally inertial frames. ($g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu} + \delta^2 g_{\mu\nu}$ and $\delta g_{\mu\nu} \to 0$)
6. The metric determines causality. Point *elsewhere* do not have any causal connection with P. The boundary between causal and acausal region is called *light cone*.

7. The metric replaces the Euclidean dot product idea. ($D = g_{\mu\nu} U^\mu V^\nu$)
8. The metric is not a matrix, although it does have indices and inverse. $g^{\mu\nu}$ is defined by $g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma$

The metric tensor for flat space-time in Cartesian co-ordinate system is:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{2}$$

where $\eta_{\mu\nu}$ are the components of Minkowski metric. For flat space-time in spherical polar co-ordinate system, the metric takes the following form.

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{3}$$

On the z-axis, $\phi = 0$. We define a basis vector: $d\phi = V^\mu = (0, 0, 0, 1)$.

$$g_{\mu\nu} V^\mu V^\nu = r^2 \sin^2\theta \tag{4}$$

So the normalized basis vector will be $U = \frac{1}{r\sin\theta} d\phi = (0, 0, 0, \frac{1}{r\sin\theta})$, so that $g_{\mu\nu} U^\mu U^\nu = 1$. The spherical polar co-ordinate system is not manifestly flat unlike Cartesian co-ordinate system.
The *geodesic equation* is the equation of the trajectory followed by a free particle. The general form of geodesic equation is

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\tau} \frac{dx^\nu}{d\lambda} \frac{dx^\tau}{d\lambda} = 0 \tag{5}$$

where λ is the affine parameter. For $g_{\mu\nu} = \eta_{\mu\nu} = (-1, 1, 1, 1)$, we have $\Gamma^\mu_{\nu\tau} = 0$. So

$$\frac{d^2 x^\mu}{d\lambda^2} = 0 \tag{6}$$

$$\Rightarrow x^\mu = a^\mu \lambda + b^\mu \tag{7}$$

For spherical polar co-ordinates, we can avoid the calculation of the connections by invoking Lagrange's equation and choosing λ to be the co-ordinate time. The equations of motion are:

$$-\frac{d^2 r}{dt^2} + r(\frac{d\theta}{dt})^2 + r\sin^2\theta(\frac{d\phi}{dt})^2 = 0 \tag{8}$$

$$-\frac{d}{dt}(r^2 \frac{d\theta}{dt}) + r^2 \sin\theta\cos\theta(\frac{d\phi}{dt})^2 = 0 \tag{9}$$

$$-\frac{d}{dt}(r^2 \sin^2\theta \frac{d\phi}{dt}) = 0 \tag{10}$$

The last equation implies the conservation of the z-component of angular momentum. Simplifying the above equations and comparing with the form of the geodesic equation (Eqn 5), we can determine $\Gamma^\mu_{\nu\tau}$.
Now we will mention one more metric which is of great interest in cosmology.

$$ds^2 = -c^2 dt^2 + t^{2p}(dx^2 + dy^2 + dz^2) \tag{11}$$

The space-time described by this metric is a good model for flat expanding universe in cosmology. $p = \frac{1}{2}$ describes radiation dominated and $p = \frac{2}{3}$ describes matter dominated universe.

Locally Inertial Frame

At each point in a curved space-time, we can construct a locally inertial co-ordinate system, which is effectively flat in a very small region about that point. We want to make a co-ordinate transformation $x^\mu \to x^{\mu'}$, so that, the metric $(g_{\mu\nu})$ becomes flat in the neighborhood of some point P.

$$g_{\mu'\nu'}(x'_P + \delta x') = g_{\mu'\nu'}(x'_P) + \left(\frac{\partial g_{\mu'\nu'}}{\partial x^{\lambda'}}\right)_P \delta x^{\lambda'} + \frac{1}{2}\left(\frac{\partial^2 g_{\mu'\nu'}}{\partial x^{\lambda'}\partial x^{\tau'}}\right)_P \delta x^{\lambda'}\delta x^{\tau'} + ... \tag{12}$$

For the co-ordinate transformation we have

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} \tag{13}$$

$$\frac{\partial g_{\mu'\nu'}}{\partial x^{\lambda'}} = \frac{\partial^2 x^\mu}{\partial x^{\lambda'}\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} + ... \tag{14}$$

$$\frac{\partial^2 g_{\mu'\nu'}}{\partial x^{\tau'}\partial x^{\lambda'}} = \frac{\partial^3 x^\mu}{\partial x^{\tau'}\partial x^{\lambda'}\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} + ... \tag{15}$$

For $g_{\mu'\nu'}(x_P) = \eta_{\mu'\nu'}$, we have 10 equations and 16 unknowns $\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)$. So we have six degrees of freedom to make the metric flat exactly at P. Now to make the first derivative of the metric vanish, we have $\frac{\partial g_{\mu'\nu'}}{\partial x^{\lambda'}} = 0$, i.e. we have 4x10 = 40 equations and 40 unknowns $\left(\frac{\partial^2 x^\mu}{\partial x^{\lambda'}\partial x^{\mu'}}\right)$. So all the first derivatives can be made to vanish. For the second derivatives to vanish, $\frac{\partial^2 g_{\mu'\nu'}}{\partial x^{\lambda'}\partial x^{\tau'}} = 0$, i.e. we have 10x10 = 100 equations and 4x20 = 80 unknowns $\left(\frac{\partial^3 x^\mu}{\partial x^{\lambda'}\partial x^{\mu'}\partial x^{\tau'}}\right)$. Therefore, in general, we cannot make all the second derivatives vanish. Actually we will get 100-80 = 20 conditions to make all the second derivatives vanish which are precisely the conditions for vanishing of curvature tensor.