Properties of Curvature

The stress tensor is the variation with the “matter action” with respect to the metric: $\frac{\delta I_m}{\delta g}$

Recall our definition of the Riemann Tensor in a set of coordinates:

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

Another form we use a lot is to lower one index:

$$R_{\rho\sigma\mu\nu} = g_{\rho\tau} R^{\tau}_{\sigma\mu\nu}$$

For a locally flat coordinate system, Carroll uses the notation: $R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}}$

The gammas vanish and the metric is flat. Therefore it has to be Cartesian.

Thus, for any Cartesian coordinate system:

$$R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}} = \frac{1}{2}(\partial_{\hat{\mu}}\partial_{\hat{\sigma}} g_{\hat{\rho}\hat{\nu}} - \partial_{\hat{\mu}}\partial_{\hat{\rho}} g_{\hat{\nu}\hat{\sigma}} - \partial_{\hat{\nu}}\partial_{\hat{\sigma}} g_{\hat{\rho}\hat{\mu}} + \partial_{\hat{\nu}}\partial_{\hat{\rho}} g_{\hat{\mu}\hat{\sigma}})$$

If we interchange mu and nu, we can see that the signs of the terms flip, therefore the symmetric part:

$$R_{\rho\sigma(\mu\nu)} = 0$$

Similarly for the hatted version:

$$R_{\hat{\rho}\hat{\sigma}(\hat{\mu}\hat{\nu})} = 0$$

What happens if we interchange sigma and rho?

$$R_{(\hat{\rho}\hat{\sigma})\hat{\mu}\hat{\nu}} = 0$$

These relations should be true in any coordinate system:

$$R_{(\rho\sigma)\mu\nu} = 0$$

Now, by looking at the second derivatives of the metric and their respective signs, we can see:

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$$

Cyclically reordering these, we notice by sign changes that:

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$$

Also, for the anti-symmetric parts:

$$R_{\rho[\sigma\mu\nu]} = 0$$

The conditions above also lead to:

$$R_{[\rho\sigma\mu\nu]} = 0$$

The remaining n-independent degrees of freedom is not $n^4$ but rather

$$N_n = \frac{n^2(n^2-1)}{12}$$ ( n = # of dimensions)

So for:

$n = 2 \qquad N_2 = 1$

$n = 3 \qquad N_3 = 6$

$n = 4 \qquad N_4 = 20$

Next, we wish to look at: $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu}$. In this local coordinate system:

$$\nabla_{[\hat{\lambda}} R_{\hat{\rho}\hat{\sigma}]\hat{\mu}\hat{\nu}} = 0$$

This is called the Bianchi Identity.

This is similar to the Jacobi Identity:

$$[[\nabla_\mu, \nabla_\nu], \nabla_\lambda] + [[\nabla_\nu, \nabla_\lambda], \nabla_\mu] + [[\nabla_\lambda, \nabla_\mu], \nabla_\nu] = 0$$

Suppose we contract $\Gamma^\rho_{\mu\nu}$ with $g_{\sigma\rho}$:

$$\Gamma_{\sigma\mu\nu} = g_{\sigma\rho}\Gamma^\rho_{\mu\nu}$$

$$= \frac{1}{2}[\frac{\partial g_{\sigma\mu}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}]$$

We can define the Riemann Tensor with index down in terms of these partial derivatives.

Next, let's look at the symmetric and antisymmetric part. Take the Riemann Tensor and Contract the 1$^{st}$ and 3$^{rd}$:

$$R_{\sigma\nu} = R^\rho_{\sigma\rho\nu} = R_{\nu\sigma}$$

This is symmetric. It is called the Ricci Tensor.

Another contraction, called the Ricci Scalar, is:

$$R = R^{\mu}_{\mu} = R^{\mu\nu} g_{\mu\nu}$$

We define the completely anti-symmetric part to be:

$$C^{\rho}_{\sigma\mu\nu} = R^{\rho}_{\sigma\mu\nu} - \frac{2}{n-2}(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}) + \frac{2}{(n-1)(n-2)} g_{\rho[\mu}g_{\nu]\sigma}R$$

Subtract all the parts with contractions on $\mu\nu$ in C.

All possible contractions are zero. It has all of the symmetries of $R^{\rho}_{\sigma\mu\nu}$.

Next, look at the following contraction:

$$g^{\sigma\nu} g^{\lambda\mu} \nabla_{\lambda} R_{\rho\sigma\mu\nu} \Rightarrow \nabla^{\mu} R_{\rho\mu} = \frac{1}{2} \nabla_{\rho} R$$

For,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Then,

$$\nabla^{\mu} G_{\mu\nu} = 0$$

Where $G_{\mu\nu}$ is the Einstein Tensor.

The Einstein Equation will be:

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

Also,

$$\nabla^{\mu}(G_{\mu\nu} = 8\pi G_N T_{\mu\nu})$$

Where each side of the equation equals 0.

The Action Principle for Gravity is:

$$I_G = \frac{1}{8\pi G_N} \int R\sqrt{-g}\; d^4x$$

Recall that we've written the Action Principle for a point particle:

$$\frac{1}{2}m \int g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} d\tau$$

In order to be able to couple this to the Einstein Equations, we have to add two pieces:

$$I_{\texttt{point particle}} = \frac{1}{2}m \int g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} \frac{\delta^4(z^{\mu}(\tau) - x^{\mu})}{\sqrt{-g}} d\tau$$

Find $x^{\mu}$ that solves the Geodesic Equation.

Solve for $x^{\mu}(\tau)$. Then this has support when the orbit we choose is the geodesic.

$$f(x_0) = \int f(x)\delta(x - x_0)\; dx$$

(True no matter what the volume element is)