

# General Relativity: Lecture 2

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In the first lecture we talked about the properties of two different types of transformations, rotations and Lorentz boosts. Their form in two dimensions were written as:

$$x' = \cos(\theta)x + \sin(\theta)y \quad (1)$$

$$y' = \cos(\theta)y - \sin(\theta)x \quad (2)$$

for the rotation by angle  $\theta$ . The boost was written in the x-t plane as:

$$x' = \gamma(x - vt) \quad (3)$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \quad (4)$$

for a boost by velocity  $v$ , with  $\gamma = \left(\sqrt{1 - \frac{v^2}{c^2}}\right)^{-1}$ . These two had several common aspects, and each had a few unique features. Several are listed in the table below:

Rotation	Boost
Geometric	Geometric
Length Preserving	Proper Length Preserving
Orthogonal Matrix	Not Applicable
Coordinate Transformation	Coordinate Transformation

We here pause to think about what it means to say that a transformation is geometric. In both cases there is something unchanged which is a notion of distance. So while the coordinates are different, we say that the transformations are geometric because there is an underlying structure which is unchanged by the transformation. We will later characterize this structure with the spacetime metric.

We will now focus on the various objects that these and other coordinate transformations can act upon by beginning with a familiar object, the vector. Let us make precise our notion of what a vector is, then see how it transforms under general coordinate transformations. A vector  $\vec{v}$  has components in a given coordinate basis  $\hat{e}_\mu$  which we write as:

$$\vec{v} = v^\mu \hat{e}_\mu \quad (5)$$

Under a general change of coordinates the components of  $\vec{v}$  will change as:

$$v^{\mu'} = \frac{dx^{\mu'}}{dx^{\mu}} v^{\mu} \quad (6)$$

The prototypical example of a vector is the tangent to a curve. If the curve is given by  $P(x,y|\tau)$  then we can define the tangent by the familiar expression:  $\frac{dx^{\mu}}{d\tau} = v^{\mu}$ . Under a coordinate transformation we may apply the chain rule to change  $x^{\mu}$  and we see that

$$v^{\mu'} = \frac{dx^{\mu'}}{dx^{\mu}} \frac{dx^{\mu}}{d\tau} \quad (7)$$

The equation above is the same as our original equation for the transformation properties of a vector, which is what we would have naively expected. The situation quickly becomes more complex however when we consider another example of what is commonly called a vector, the gradient.

A gradient is a useful tool to physicists and it can be constructed from the derivatives of a scalar field (e.g. Temperature). For a scalar  $f(x,y)$  the gradient may be defined by:

$$\vec{\nabla} f = \frac{\partial f}{\partial x^{\mu}} \quad (8)$$

Under a transformation we see that the gradient of  $f(x,y)$  does not transform the same way as the tangent, which we had called a vector. In particular, by application of the chain rule again, we see that the gradient of a scalar transforms as:

$$\frac{\partial f}{\partial x^{\mu'}} = \frac{dx^{\mu}}{dx^{\mu'}} \frac{\partial f}{\partial x^{\mu}} \quad (9)$$

So by looking at the transformation properties of two objects which we originally thought were both vectors, we find that the tangent to a curve and the gradient of a scalar are two different objects. We call tangent vectors, and objects which behave the same way under transformation contravariant vectors. We will call gradients and other objects with similar transformation rules covariant vectors. We will write a covariant vector as:

$$\mathbf{X} = \vec{X} = X_{\mu} \hat{e}^{\mu} \quad (10)$$

Now, in order to do arithmetic with these newly found contravariant and covariant vectors we need to consider how we will take a vector product. The two most well known vector products are the dot product and the cross product written as  $\vec{V} \cdot \vec{W}$  and  $\vec{V} \times \vec{W}$  respectively where  $\vec{V} = V^{\mu}$  and  $\vec{W} = W^{\nu}$ . In order to make the dot product appear as a scalar, and for the cross product to appear as a vector we will once again need new tools.

To make the cross product appear as a vector we will introduce the Levi-Civita symbol, a completely antisymmetric object written as  $\epsilon_{\mu\nu\gamma}$ . So we may write a cross product as:

$$\vec{V} \times \vec{W} = \epsilon_{\mu\nu\gamma} V^\mu W^\nu = X_\gamma \quad (11)$$

As an added bonus this same object makes the scalar tripple product into a scalar without us introducing anything new. We can see this since

$$\vec{U} \cdot (\vec{V} \times \vec{W}) = U^\gamma (\epsilon_{\gamma\mu\nu} V^\mu W^\nu) = U^\gamma X_\gamma \quad (12)$$

where the right most expression is a scalar by the Einstein summation convention.

Now to make the dot product appear explicitly as a scalar we need to use the metric which will be called  $g_{\mu\nu}$ . The metric as it is written can lower the index on a vector  $g_{\mu\nu} V^\nu = V_\mu$ , or in other words it maps contravariant vectors to covariant vectors. Using this we will write a dot product as:

$$\vec{V} \cdot \vec{W} = g_{\mu\nu} V^\mu W^\nu \quad (13)$$

What the metric gives us is the ability to talk about measurable quantities. In the two cases we considered earlier of rotations and lorentz boosts, the physical quantity that was unchanged was a type of length which as we know can be written as a dot product. As a side note, even though the metric structurally is a square array it lacks all the complete set of properties of a matrix. Lastly, we can define an inverse metric which raises indicies. We will call it  $g^{\mu\nu}$  and define it by

$$g^{\mu\nu} g_{\nu\sigma} = \delta_\sigma^\mu \quad (14)$$

It is through the metric that we will learn more about the geometric structures which underlie the coordinate transformations we have been discussing, and we will spend a good part of the semester looking at the metric and associated objects.