GR NOTES

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1. Reading List and Projects

So i've been contacted by around 3 people with reading lists for their projects, I think it pertinent to give you some good complimentary material, specifically the HEP Inspire website. It has a huge archive of papers that will be helpful in you guys getting your reference materials.. Just search for the relevant keywords. Also we have a huge number of resources at the website ArXiv.org or whatever. You can search by author or by keyword but the point is they often will not have the list of references linked while you can probably go into Inspire, find the same paper, and find the links to the other reference papers.

2. class

Last time we ended with the promise to talk about tensors. As you recall we had 1-index components of objects we called vectors and co-vectors or one forms, we wrote vector components as V^μ and one-form or co-vector components as W_μ. Really we have vectors and tensors as operators, yet we won't be too concerned with this notation since we will usually only right out the scalars corresponding to the components or indices in a certain basis, so really what we care about are the transformation properties.

We will define a <u>tensor</u> as a multi-linear map from vectors and covectors (or with other operators by contraction) to $\mathbb{R}$. We will define the tensor rank or it's order as the number of vectors and covectors it must act on to produce a scalar. If we have a (1,1) tensor or a matrix, it requires one vector and one covector. Whence if it only acts on the vector (one-form) we will be left with an object that needs a one-form (vector) to produce a scalar, or again will just fall into the category of vectors (one forms), for example $v^\mu = A^\mu_\nu u^\nu$ acts as the components of a new vector v that can be seen as the transformed components from the vector u's original components u^μ. For higher order tensors, we can say a tensor of order (k,l) maps k one-forms and l vectors multi linearly, so it must transform as a combination of k vectors (for the k 1-forms it maps) and l covectors/one-forms under changes of coordinate system or other transformations. In general we may write this property as, for a tensor T of order (k,l) at a point p in the manifold

$$T : \underbrace{\mathbb{T}_p^* \times \mathbb{T}_p^* \times \cdots \times \mathbb{T}_p^*}_{\text{k times}} \times \overbrace{\mathbb{T}_p \times \mathbb{T}_p \times \cdots \times \mathbb{T}_p \times}^{\text{l times}} \longrightarrow \mathbb{R}$$

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Where $\mathbb{T}_p$ is the tangent space at p in the manifold (where the vectors live) and $\mathbb{T}_p^*$ is the cotangent space, where the one-forms or covectors (or dual vectors) live. Technically, the structure of this is that this mapping is more specifically a ring homomorphism from

$$T : \underbrace{\mathbb{T}_p^* \otimes \mathbb{T}_p^* \otimes \cdots \otimes \mathbb{T}_p^*}_{\text{k times}} \otimes \overbrace{\mathbb{T}_p \otimes \mathbb{T}_p \otimes \cdots \otimes \mathbb{T}_p \otimes}^{\text{l times}} \longrightarrow \mathbb{R}$$

where the $\otimes$ stands for the multilinear tensor product, which gives the tensor the structure of a member of the dual space of this tensor product space, or being in other words the sum of elements with components having the form $V_1 \otimes V_2 \otimes \cdots V_k \otimes U^1 \otimes U^2 \cdots \otimes U^l$ where $V_i e^i$ are elements of the dual or cotangent space and $U^j e_j$ are the elements of the tangent vector space. The tensor product, being bilinear, has the properties:

$$(av_1 + bu_1) \otimes (cv_2 + du_2) = (ac)(v_1 \otimes v_2) + (ad)(v_1 \otimes u_2) + (bc)(u_1 \otimes v_2) + (bd)(u_1 \otimes u_2)$$

$\forall v_1, v_2, u_1, u_2$ in our vector space and all scalars $a, b, c, d \in \mathbb{R}$. This can be extrapolated to higher tensor products.

Now we have several types of operations between tensors, one is contraction where the tensors act on the indices of the other tensor to map them to $\mathbb{R}$, which for tensors with components in a certain basis $T_{\mu\cdots\alpha}{}^{\nu\cdots\gamma}$ and $A_{\mu\cdots\alpha}{}^{\nu\cdots\gamma}$ we could contract like indices to get a scalar for that induce

$$T_{\mu\cdots\alpha}{}^{\nu\cdots\gamma} A_{\gamma\cdots\beta}{}^{\alpha\cdots\lambda}$$

where we add by einstein notation and reduce the rank we see by one covector (covariant) index and one vector (contravariant) index. We can also form an outer product $T \otimes A$ with components $T_{\mu\cdots\alpha}{}^{\nu\cdots\gamma} A_{\zeta\cdots\beta}{}^{\theta\cdots\lambda}$ which we see as an operator acting on the same number of vectors and convectors as both tensors did separately. We have already been dealing with one tensor, the metric tensor which we had

$$l(v) = g_{\mu\nu} v^\mu v^\nu$$

or in other words this is a rank two tensor that acts on vectors to produce the length of the vector. However remember, contracting a tensor only partially leaves the rest of it's indices still as objects, which means we could act on a single vector with g and be given back something that still needs a vector to produce a scalar, or in other words $g(v, -)$ where only v has been acted on is actually a one form. In general the corresponding one form produced specifically with the contraction of the metric is given the same symbol, <u>even though it is now a one form</u>. This can be written as an object with components

$$v_\mu = g_{\mu\nu} v^\nu$$

Similarly using the inverse of g, the contravarient formulation, we see

$$v^\mu = \delta^\mu_\nu v^\nu = g^{\mu\gamma} g_{\gamma\nu} v^\nu = g^{\mu\gamma}(g_{\gamma\nu} v^\nu)$$

$$= g^{\mu\gamma} v_\gamma$$

or in other words we see we have the same relation, we create a vector from a covector by acting on it with the inverse of the metric to get vector or contravarient form of the operator with contravarient indices

$$w^{\nu} = g^{\nu\mu} w_{\mu}$$

where w_{μ} were the μ'th coefficients of the one-form/covector w.

We can now examine a class of tensors that g our metric tensor fell into. We noticed that

$$g_{\mu\nu} w^{\mu} v^{\nu} = w \cdot v = v \cdot w = g_{\mu\nu} v^{\mu} w^{\nu} = g_{\nu\mu} w^{\mu} v^{\nu}$$

$$\implies (g_{\mu\nu} - g_{\nu\mu}) w^{\mu} v^{\nu} = 0 \qquad \text{for any vectors } v \text{ and } w$$

or in other words $g_{\nu\mu} = g_{\mu\nu}$. So the metric is symmetric in it's indices, and for any two index tensor $T = T_{\mu\nu}$ we can write $T_{\mu\nu} = \frac{1}{2}\left[(T_{\mu\nu} + T_{\nu\mu}) + (T_{\mu\nu} - T_{\nu\mu})\right] = T_S + T_A$ or the symmetric and antisymmetric portion. If the tensor is symmetric, as g is, the second portion is just zero, while if it is anti-symmetric then the first part is zero.
We can generalize this to the symmetric tensors and antisymmetric tensors, where the symmetric form of a tensor is displayed as

$$T_{(\mu\cdots\alpha)} = \frac{1}{n!} \sum_{\text{all perm.}\sigma} T_{\sigma(\mu\cdots\alpha)}$$

while the antisymmetric case is written as

$$T_{[\mu\cdots\alpha]} = \frac{1}{n!} \sum_{\text{all perm.}\sigma} sgn(\sigma) * T_{\sigma(\mu\cdots\alpha)}$$

Specifically for the rank 3 case we have

$$T_S = T_{(\mu\nu\gamma)} = \frac{1}{6}\left(T_{\mu\nu\gamma} + T_{\mu\gamma\nu} + T_{\nu\mu\gamma} + T_{\nu\gamma\mu} + T_{\gamma\mu\nu} + T_{\gamma\nu\mu}\right)$$

while

$$T_A = T_{[\mu\nu\gamma]} = \frac{1}{6}\left(T_{\mu\nu\gamma} - T_{\mu\gamma\nu} - T_{\nu\mu\gamma} + T_{\nu\gamma\mu} + T_{\gamma\mu\nu} - T_{\gamma\nu\mu}\right)$$