PHZ 6607: General Relativity Notes, 9/17/2014-9/23/2014
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Maxwell's Equations can be written:

$$\left.\begin{array}{c} \nabla_i B^i = 0 \\ \nabla_i E^i = 4\pi\rho \\ \epsilon^{ijk}\nabla_j E_k = -\dot{B}^i \\ \epsilon^{ijk}\nabla_j B_k = \dot{E}^i + 4\pi J^i \end{array}\right\} \nabla_j \left(T_i^{\ j} = -E_i E^j - B_i B^j + \frac{1}{2}\delta_i^{\ j}(E_k E^k + B_k B^k) \right) = 0$$

if the symbol, ∇_i, is defined as the covariant derivative:

$$\nabla_i = \partial_i + \Gamma_i^{\ \lambda}{}_{\lambda}$$

where ∂_i is the usual partial derivative and Γ_i is a connection term. This is necessary to maintain tensor transformations. But we still need to introduce a way to raise and lower the indices. We can do this with the relations between the metrics, g^{ij} and g_{ij}, and by definition,

$$\nabla_k g_{ij} = 0$$

The presence of the metric must be built into the compatibility of Maxwell's Equations as covariant equations. Being covariant is to be free of coordinate relations. The equations vary in the same sense as the coordinates. Any physical system should be able to be described by any coordinate system. What matters is the metric that links the coordinates together, and how this relates to the change in coordinates. This is a simple explanation for why most laws of physics are differential equations. We care about the changes in coordinates, not the coordinates themselves. In order for any equation to be universal (equivalent in any coordinate system), a metric is implied as well. An example can be shown with the continuity equation. The continuity equation tells us that charge is conserved; therefore, it should be equivalent in any coordinate system. It is described by the face that the divergence of the current density should be equal to the flow of charge density out of the system. Written covariantly,

$$\nabla_\mu J^\mu = 0 = \frac{1}{\sqrt{-\mathbb{g}}}\partial_\mu \sqrt{(-\mathbb{g})} J^\mu$$

Understanding of the non-relativistic physical laws and intuition allows us to guess that the concept of moving charges generating an electromagnetic force could be analogous to moving mass (and therefore energy) generating a gravitational force.

<u>Gravitational vs. Inertial Mass:</u>
We see many similarities between Maxwell's equations and the non-relativistic equations of Newton's gravity. Comparing the mass that arises from the laws of gravitation with the mass that comes from Newtonian kinematics due to an electric field,

Electromagnetic Force

The electric field can be defined as the negative gradient of a potential, and the force follows:

$$\vec{E} = -\vec{\nabla}\phi_E$$
$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) = m_i\vec{a}$$

Written covariantly,

$$F_{\mu\nu} = 2\nabla_{\{\mu}A_{\mu]} \rightarrow \nabla_\mu F^{\mu\nu} = J^\nu$$
$$\nabla_\nu\nabla_\mu F^{\mu\nu} = \nabla_\nu J^\nu = 0$$

Where the second relation follows from Gauge Invariance and can be identified as a conservation equation.

Gravitational Force:

Likewise, the gravitational field can be defined as the negative gradient of a potential, and the force follows:

$$\vec{a}_G = -\vec{\nabla}\phi_G$$
$$\vec{F}_G = m_G\vec{a}_G$$

In relativity, we relate this to the curvature and find:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left(R^{\sigma\rho}g_{\sigma\rho}\right) = 8\pi G_N T_{\mu\nu}$$

To proceed, we must consider the thought problem of an accelerating frame of reference, and a frame of reference under the influence of an external gravitational field. It is impossible to distinguish an accelerating frame from a frame in the presence of an external gravitational field. This leads us to two principles.

Weak Equivalence Principle- This principle relates the equivalence of the inertial mass, m_i, to the gravitational mass, m_g. Every object will accelerate the same in the same gravitational potential if the gravitational mass and inertial mass are equivalent, which has so far been observed as true. Extrapolating this implies that one cannot distinguish between a test particle accelerating due to an external gravitational force from an accelerating frame in a small enough region of spacetime. When we say a small enough region of space, we mean a small enough region such that the accelerating due to an external gravitational force is approximately constant instead of the $1/r^2$ form.

Einstein Equivalence Principle- Taking the Weak Equivalence Principle a step further, we know from Special Relativity that mass is another type of energy. So all laws of physics (including gravity) should reduce to those of special relativity in a small enough region of space.

We propose that the curvature of the trajectory of a particle will be due to space being curved, not an external gravitational field. This can be summed up saying it is impossible to detect gravitational locally (distinguish gravitation from bending spacetime). To describe effects like this, we must work with mathematical constructs called manifolds.

Differentiable Manifolds

It is easy to give examples of manifolds, but taxing to define them exactly. A manifold is an object used to describe a space that may be curved or have a complicated topology, but can locally be described by Euclidean $\mathbb{R}^n$. Many manifolds can be sewn smoothly together to reconcile with the original space, smoothly

meaning differentiable. To fully define a differentiable manifold, we need to define other terms that will be incorporated.

Map- a map, ϕ, is something that relates a set $M \rightarrow N$, meaning one element of M is assigned one element of N. This is a generalization of a function.

-A map is one-to-one (injective) if each element of a set N has exactly one element of M mapped into it.

-A map is onto (subjective) if each element of N has at least one element of M mapped into it.

Because we care about differentiability, we denote a map as C^n if its nth derivative exists. We will mostly deal with C^∞.

-Two sets are diffeomorphic if there exists a C^∞ map between them with a C^∞ inverse. This is generally what we think about when we think two spaces are equal.

Chart- Consists of a subset U of a set M that is mapped into $\mathbb{R}^n$. This is a generalized idea of a coordinate system.

Atlas- an indexed collection of charts $(U_1 \dots U_k)$ that satisfies:

1) The union of all subsets of M covers the entire set: $U \dots U_\alpha = M$

2) The charts are smoothly sewn together. If the intersection of $U \dots U_\alpha \neq 0$, then $U_\alpha \wedge U_\beta$ is smooth.

Combining all these ideas, a manifold can be defined as a set M with a maximal atlas, containing every possible compatible chart. So it is a continuous space with a coordinate system that locally looks like $\mathbb{R}^n$. This gives us a basic definition for a tool that we can manipulate for future discussions. If we are to use the idea of a manifold, we need to redefine our other tools to suit the purpose of a manifold.

Vectors (Contravariant) on Manifolds:

Using the notion of manifolds, and expecting to use them later on, it is logical to begin defining structures on manifolds; vectors being the ideal starting point. Due to the intricacies of manifolds, we should abandon the idea that vectors exist from one point to another on the manifold. The idea of a vector should exist at a single point, not relying on neighboring points as a definition. In addition, the idea of a vector should be independent of coordinates. To remove the idea of stretching from one point to another, one might be inclined to adopt the idea of a directional derivative. Directional derivatives exist at single points, and point in a general direction. Considering a point, P, on a manifold, and all possible curves passing though this point, parameterized by some variable, λ. If we differentiate some function, f, with respect to the parameter, λ, this give us the concept of a differential, but what about the directional part? This should be independent of coordinates for this concept to mean anything in the context of General Relativity, and yet coordinates should be readily applied for us to use these ideas. We can easily include these two concepts if we evoke the chain rule, and relate the parameterized curves to the coordinate system we place on the manifold.

$$\frac{d}{d\lambda}f = \frac{dx^\mu}{d\lambda}\left(\frac{\partial f}{\partial x^\mu}\right) \quad \rightarrow \quad \frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu$$

The components of the vector in the coordinate system are given by $\frac{dx^\mu}{d\lambda}$, while the direction is given by ∂_μ. It is now fruitful to first define a tangent space, T_P, as being the set of all vectors at a single point in spacetime. This will be useful for future discussions. We can extend the idea of a vector at a single point to the whole manifold with the vector field. Vector fields define a map from smooth functions to smooth functions all over the manifold; it is the collection of all vectors at all points. Vector components transform as the usual,

$$V^\mu \rightarrow V^{\mu'} = \Lambda^{\mu'}_{\nu} V^\nu$$

It is also useful to define the commutator, also called a Lie Bracket,

$$[U, V] = UV - VU$$

If we define the operators $V(f) = \frac{df}{d\lambda}$, and $U = \frac{d}{d\eta}$, then the commutator of these two operators acting on a function, f, would be

$$[U, V]f = U\big(V(f)\big) - V\big(U(f)\big) = \frac{d}{d\eta}\left(\frac{df}{d\lambda}\right) - \frac{d}{d\lambda}\left(\frac{df}{d\eta}\right)$$

This is coordinate independent, and will be useful for future discussions.

Covectors (One-Forms, Dual Vectors, Covariant):

Following the same logic as above with vectors, we can define an object called a covector, an object to turn a vector on a manifold into a pure scalar in $\mathbb{R}^n$. Extrapolating this to a vector field, the action of a covector field on a vector field should yield a function. We set up a dual vector space such that the dual space to the tangent space is the cotangent space, T_P^*. The dual space is the space of all linear maps from the vector space to $\mathbb{R}^n$. Because it is arbitrary whether we start with a vector or covector, the covector space clearly must have the same dimensions as the vector space, and itself forms a vector space. Considering some curve, f, on a manifold, taking its derivative yields,

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu \rightarrow d = dx^\mu \frac{\partial}{\partial x^\mu}$$

The term $\frac{\partial f}{\partial x^\mu}$ are the components of the function in terms of the coordinate basis used to describe this function, and the dx^μ terms are the basis elements in the space of the one-forms. Covector components transform under the same principle as vectors

$$\omega_\mu \rightarrow \omega_{\mu'} = \Lambda^{\nu}_{\mu'} \omega_\nu$$

Tensors:

Taking the concept of structures on manifolds one step further, we can create the idea of a tensor. Tensors are multilinear maps between (k) vectors and (l) covectors to $\mathbb{R}^n$ and can be defined by a rank (k, l),

$$\mathbb{T} = T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l} \partial_{\mu_1} \otimes \ldots \partial_{\mu_k} \otimes \partial\nu^1 .. \otimes \partial\nu^l$$

The tensor space is spanned by all vector and covector basis composing the tensor. Extrapolating this to how tensors transform, we can deduce that they will transform as the combination of vectors and covectors transform,

$$T^{\mu'_1 \dots \mu'_k}_{\nu'_1 \dots \nu_{l'}} = \Lambda^{\mu'_1}_{\mu_1} \dots \Lambda^{\mu'_k}_{\mu_k} \Lambda^{\nu_1}_{\nu'_1} \dots \Lambda^{\nu_l}_{\nu'_l} T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}$$