General Relativity Notes: Sept. 9, 2014

Andy Chilton

1. LORENTZ TRANSFORMS

Recall that, for a boost in the x-direction, the Lorentz Transform of the coordinates x and t are given by

$$\begin{aligned} x' &= \gamma\,(x - vt) = \gamma\left(x - \frac{v}{c}ct\right) && (1) \\ t' &= \frac{\gamma}{c}\left(ct - x\frac{v}{c}\right) && (2) \end{aligned}$$

where $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$. We'd like to express this in terms of the matrix elements of the thing we discussed previously, $\Lambda^{\mu'}_{\mu}$, which transforms vectors according to

$$X^{\mu'} = \Lambda^{\mu'}_{\mu} X^{\mu} \tag{3}$$

It's easy enough to make this decomposition:

$$\Lambda = \gamma \begin{pmatrix} 1 & -v/c & 0 & 0 \\ -v/c & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

We can also parametrize using $\tanh(\alpha) = v/c = \beta$:

$$\Lambda = \gamma \begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) & 0 & 0 \\ -\sinh(\alpha) & \cosh(\alpha) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5}$$

2. SOME COMMENTS ON LINE ELEMENTS, VOLUME ELEMENTS, AND THE METRIC

Let's look at the Λ's in polar coordinates. In two dimensions, the transformation to polar coordinates from Cartesian coordinates is given by

$$\begin{aligned} x &= r\cos(\theta) && (6) \\ y &= r\sin(\theta) && (7) \end{aligned}$$

and the differentials

$$\begin{aligned} dx &= dr\cos(\theta) - r\sin(\theta)d\theta && (8) \\ dy &= dr\sin(\theta) + r\cos(\theta)d\theta && (9) \end{aligned}$$

In matrix form, this is

$$dX^{\mu'} = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} dX^{\mu} \tag{10}$$

Here the μ' coordinate system is polar, μ is Cartesian, and the matrix represents Λ. The line element is

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2 \tag{11}$$

The volume element is

$$dV = dx * dy = r * dr * d\theta \tag{12}$$

The ratio of the coefficients (= r) is actually equal to the determinant of Λ. Cool.
Now let's look at how the metric transforms:

$$\begin{aligned} g_{\mu'\nu'} &= \Lambda^{\mu}_{\mu'} \Lambda^{\nu}_{\nu'} g_{\mu\nu} && (13) \\ \implies \mathrm{Det}(g_{\mu'\nu'}) &= \mathrm{Det}(\Lambda)^2 \mathrm{Det}(g_{\mu\nu}) && (14) \\ \implies \mathrm{Det}(\Lambda) &= \left(\frac{\mathrm{Det}(g_{\mu'\nu'})}{\mathrm{Det}(g_{\mu\nu})} \right)^{1/2} && (15) \end{aligned}$$

If $g_{\mu\nu}$ is in Cartesian coordinates, then this reduces to

$$\mathrm{Det}(\Lambda) = \left(\mathrm{Det}(g_{\mu'\nu'})\right)^{1/2} \tag{16}$$

Generalizing to N dimensions, and using the information we gained earlier,

$$dV = \left(\pm \mathrm{Det}(g_{\mu\nu})\right)^{1/2} dx_1 * dx_2 * ... * dx_N \tag{17}$$

The $\pm$ inside the square root is to take care of the signature of the metric; if the signature (i.e. product of its eigenvalues) is negative, then use the negative sign, or vice versa. If we didn't do this, a Lorentzian metric would yield an imaginary volume element, which is nonsensical.

3. SIDE NOTE FROM PREVIOUS CLASS: CAROLL'S TIME DILATION CONSTRUCTION

We were supposed to be comparing PROPER times in this circumstance to show that τ_{abc} ¡ τ_{ac} for all velocities; Caroll apparently used t rather than τ here.

4. VECTORS

Consider a vector $\mathbf{V} = \vec{V} = V^{\mu} e_{\mu}$. Here the e_{μ} are basis vectors in some basis. There is a subtelty when dealing with making scalars out of these vectors. In three-space we would just take the dot product between two regular vectors and be done with it. However, in Minkowski space, we need things that actually live in a different space to produce them. These are the dual vectors, or one-forms:

$$\mathbf{W} = W_{\mu} \Theta^{\mu} \tag{18}$$

where the Θ^{μ} are basis vectors in the dual space. They and those from the original vector space obey the following relation:

$$\Theta^{\mu} e_{\nu} = \delta^{\mu}_{\nu} \tag{19}$$

5. COMMENTS ABOUT THE ORIGIN OF VECTORS

Looking at the vector $\frac{dx^\mu}{d\tau}$, which we call velocity. Normally we'd think of this guy being defined using the same coordinate basis as x^μ. But coordinates describe space; while the position vector x^μ "lives" in space, velocity really doesn't, so it makes less sense to describe in terms of those. Furthermore, vectors are truly geometrical objects and should be able to be defined separate from any particular coordinate system. In fact (to be fleshed out in coming classes) we will use what's called the Tangent Space T_p at a particular point on a manifold to define vectors and tensors. T_p has the same dimension as the underlying manifold, and the collection of all T_p for a particular manifold is called the Tangent Bundle.