"Conformal Field Theory" (Notes) 2-11-10

Reference:

"Conformal Field Theory" - Di Francesco P., Mathieu P., Senechal D.

"Applied Conformal Field Theory" - Paul Ginsparg arXiv:hep-th/9108028v1 11 Nov 1988

"Lectures on String theory" - D. Lüst, S. Theisen Lecture notes in Physics (Springer)

and BBS of course!

Conformal Group in d dimensions

- Suppose we are in a spacetime of dimension d with metric $g_{\mu\nu}(x)$
- We define a conformal transformation as an invertible mapping, $x \to x'$:

$$g'_{\mu\nu}(x') = \Omega(x)\, g_{\mu\nu}(x) \qquad (1)$$

as a subgroup of general transformations

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\, g_{\alpha\beta}(x) \qquad (2)$$

- Called conformal because leave the angle

$$\cos\theta = \frac{v \cdot w}{\sqrt{v^2 w^2}} \longrightarrow \frac{\Omega(x)\, v \cdot w}{\sqrt{\Omega^2(x) v^2 w^2}} = \cos\theta \quad \text{invariant!}$$

- This set forms a group called the Conformal Group
  - It follows immediately that the Poincaré group is a subgroup with $\Omega = 1$!

Infinitesimal Generators

- To find the generators, consider an IT

$$x^\mu \to x^\mu + \epsilon^\mu(x)$$

Then from (2),

$$g_{\mu\nu}(x) \longrightarrow g_{\mu\nu} - (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu) + O(\varepsilon^2)$$

If we wrote

$\Omega(x) = e^{-\omega(x)}$, $g_{\mu\nu} \to (1-\omega(x))g_{\mu\nu}$

conformal transformation implies

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu \propto g_{\mu\nu} = k\, g_{\mu\nu}$$

Taking Tr of both sides, we find

$$k = \frac{2(\partial\cdot\varepsilon)}{d}$$

or for a CT:

$$\boxed{\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d}\partial_\alpha \varepsilon^\alpha g_{\mu\nu}} \qquad (3)$$

Applying ∂_ρ (3), permuting indices and taking a LC, and taking $g_{\mu\nu} = \eta_{\mu\nu}$, we find

$$2\partial_\mu \partial_\nu \varepsilon_\rho = \eta_{\mu\rho}\partial_\nu f + \eta_{\nu\rho}\partial_\mu f - \eta_{\mu\nu}\partial_\rho f$$

where $f = \frac{2}{d}\partial_\alpha \varepsilon^\alpha$

contracting with $\eta^{\mu\nu}$, we get:

$$2\partial^2 \varepsilon_\mu = (2-d)\partial_\mu f$$

Applying ∂_ν on RHS and ∂^2 on (3) we get

$$(\eta_{\mu\nu}\partial^2 + (d-2)\partial_\mu\partial_\nu)(\partial_\alpha \varepsilon^\alpha) = 0 \qquad (4)$$

- (can check by applying $\partial^\mu\partial^\nu$ to (3) and comparing to $\eta^{\mu\nu}$ (4))
  - $d=1$ imposes no constraints, $d=2$ special
  - From (4), we see ε_μ must be at most Quadratic in x & $(d \geq 3)$

Cases

- $\varepsilon^\mu = a^\mu$ Translation
- $\varepsilon^\mu = \omega^\mu_\nu x^\nu$ $\omega_{\mu\nu}$ antisymmetric Rotation
- $\varepsilon^\mu = \lambda x^\mu$, scale (Dilatation)
- $\varepsilon^\mu = b^\mu x^2 - 2x^\mu(b\cdot x)$

Special CT's

Can check this satisfies:

$$\partial_\mu \epsilon_\nu = 2 b_\nu x_\mu - 2\eta_{\mu\nu}(b\cdot x) - 2 x_\nu b_\mu$$

$\Rightarrow$

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = -4\eta_{\mu\nu}(b\cdot x)$$

while

$$\tfrac{2}{d}(\partial_\alpha \epsilon^\alpha)\eta_{\mu\nu} = \tfrac{4}{d}\eta_{\mu\nu}(b\cdot x) - \tfrac{2}{d}\eta_{\mu\nu}\left(-2d\,(b\cdot x) - \tfrac{4}{d}(b\cdot x)\right)$$

$$= -4\eta_{\mu\nu}(b\cdot x) \checkmark$$

- Can exponentiate these to get finite transformations:

$$\boxed{\begin{array}{lll} \text{Translation} : & x'^\mu = x^\mu + a^\mu \\ \text{Dilation} : & x'^\mu = \alpha x^\mu \\ \text{Rotation} : & x'^\mu = \Lambda^\mu{}_\nu x^\nu \\ \text{SCT} : & x'^\mu = \dfrac{x^\mu - b^\mu x^2}{1 + 2b\cdot x + b^2 x^2} \end{array}} \qquad (5)$$

- Can see the SCT one as follows:
  Note that the group contains the inversion element
  $x^\mu \to \frac{x^\mu}{x^2}$; Then consider:
  Inversion – Translation – Inversion

$$x^\mu \to \frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} + b^\mu \to \frac{(x^\mu/x^2 + b^\mu)}{(x^\mu/x^2 + b^\mu)^2}$$

$= (5) \checkmark$

and taking b^μ infinitesimal, we get back

$$x'^\mu = (x^\mu + b^\mu x^2)(1 - 2b\cdot x - b^2x^2) = x^\mu - 2x^\mu\, b\cdot x + b^\mu x^2$$

or $\delta x^\mu = \epsilon^\mu = b^\mu x^2 - 2x^\mu\, b\cdot x \checkmark$

In ~~total~~, have

$$\delta x^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu (b\cdot x)$$

has

$$d + \frac{d(d-1)}{2} + 1 + d = \boxed{\tfrac{1}{2}(d+2)(d+1)}$$

Generators!

then we may write generators for these transformations as:

$$\begin{cases} P_\mu = -i\partial_\mu & \text{(trans)} \\ D = -i x^\mu \partial_\mu & \text{(Dilation)} \\ L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) & \text{(Rotation)} \\ K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) & \text{(SCT)} \end{cases}$$

and generate finite by $e^{i\omega_a T^a}$ ...

- Can verify that these satisfy the algebra:

$$[D, P_\mu] = i P_\mu$$
$$[D, K_\mu] = -i K_\mu$$
$$[K_\mu, P_\nu] = 2i(\eta_{\mu\nu} D - L_{\mu\nu})$$
$$[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu)$$
$$[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu)$$

- $[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho})$

- Can make this simpler by defining $J_{ab} = -J_{ba}$

$$\begin{cases} J_{\mu\nu} = L_{\mu\nu} & J_{-1\mu} = \frac{1}{2}(P_\mu - K_\mu) \\ J_{-10} = D & J_{0\mu} = \frac{1}{2}(P_\mu + K_\mu) \end{cases}$$

$a, b = -1, 0, 1, \dots d$

which obey

$$[J_{ab}, J_{cd}] = i(\eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac})$$

where we have introduced

$\eta_{ab} = \mathrm{diag}(-1, 1, 1, \dots 1)$ if $\eta_{\mu\nu}$ is Euclidean

$\quad\;\; = \mathrm{diag}(-1, -1, 1, \dots 1)$ if $\eta_{\mu\nu}$ is Minkowski

and in general if

$$\eta_{\mu\nu} = (\underbrace{-1, -1, \dots}_{p}, \underbrace{1, 1, \dots 1}_{q})$$

then η_{ab} has 1 more negative component $(-1, \dots)$

So say Group is isomorphic to $SO(p+1, q+1)$

- Note: Poincaré + Dilations form a subgroup
  ⇒ ~~theory~~ invariant under translations, rotations and scale is not necessarily conformally invariant

§ Constraints

- N point functions that are conformally invariant are very restricted! $(\Gamma(x_i))$
- Translation + Rotation
  only $|x_i - x_j|$ dependance
- Scale
  only ratios $\frac{|x_i - x_j|}{|x_k - x_l|}$ must appear
- SCT:
  $$|x_i' - x_j'| = \frac{|x_i - x_j|}{(1 - 2b\cdot x_i + b^2 x_i^2)^{1/2}(1 - 2b\cdot x_j + b^2 x_j^2)^{1/2}}$$
  ⇒ only cross ratio's must appear! cancel factors
- Impossible to construct with only 2 or 3 points
- For 4 could have
  $$\frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_3||x_2 - x_4|} \quad \text{or} \quad \frac{|x_1 - x_2||x_3 - x_4|}{|x_2 - x_3||x_1 - x_4|}$$
- Turns out there are
  $N(N-3)\frac{1}{2}$ such independent ratio's!

§ Energy Momentum Tensor + CI!

Under an IT $x^\mu \to x^\mu + \epsilon^\mu$
with
$\delta S = 0$, then there is a conserved energy momentum tensor
$$\partial_\mu T^{\mu\nu} = 0,$$
If ϵ^μ is now $\epsilon^\mu(x)$, will have
$$\delta S = \int d^d x\, \partial_\mu (T^{\mu\nu} \epsilon_\nu)$$

$$\delta S = \int d^d x \; T^{\mu\nu} \partial_\mu \varepsilon_\nu \qquad T_{\mu\nu} = T_{\nu\mu}$$

$$= \frac{1}{2} \int d^d x \; T^{\mu\nu} (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu)$$

or

using (3):

$$\delta S = \frac{1}{d} \int d^d x \; T^\mu{}_\mu \, \partial_\rho \varepsilon^\rho \qquad (6)$$

$\Rightarrow$ if $T^\mu{}_\mu = 0 \quad \delta S = 0$

- Tracelessness of $T^{\mu\nu}$ implies conformal invariance
- Saw this was true for Bosonic String!
- Converse not true, since $\partial_\rho \varepsilon^\rho$ is not an arbitrary function.
- In some cases, a scale invariant theory can make $T^\mu{}_\mu = 0$ similar to the Belifante' method
  - See Ref 1 Ch 4 for more

Conformal Invariance in d=2 (Euclidean, $g_{\mu\nu} = \delta_{\mu\nu}$)

- As noted before, the case $d = 2$ is special:
- In two dimensions we find $\exists$ an infinite number of local conformal transformation given holomorphic mappings on some part of $\mathbb{C}$
- A subgroup of this will be a 6-parameter global conformal group

To study, we look at Eq'n (3)

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d} (\partial \cdot \varepsilon) g_{\mu\nu}$$

for $d = 2$ becomes

Taking coordinates x^1, x^2
then for $\mu = \nu = 1$

$$2\partial_1 \varepsilon_1 = (\partial_1 \varepsilon_1 + \partial_2 \varepsilon_2)(1) \qquad \partial_1 \varepsilon_1 = \partial_2 \varepsilon_2$$

while $\mu = 1 \quad \nu = 2$

$$\partial_1 \varepsilon_2 + \partial_2 \varepsilon_1 = 0 \qquad \partial_1 \varepsilon_2 = -\partial_2 \varepsilon_1$$

- If we call $x^1 = x$, $x^2 = y$ and $z = x + iy$, $\bar{z} = x - iy$

$\varepsilon(z) = \varepsilon_x + i\varepsilon_y \quad \bar{\varepsilon}(\bar{z}) = \ldots$

This is just

$$\boxed{\partial_x \varepsilon_x = \partial_y \varepsilon_y \qquad \partial_x \varepsilon_y = -\partial_y \varepsilon_x} \qquad (7)$$

Which are just the Cauchy - Reimann Eq'ns!

- So we see that 2-d CT coincide with holo (anti) - morphic coordinate transformations:

$z \to f(z) \qquad \bar{z} \to \bar{f}(\bar{z})$

- Well known that these are angle preserving mappings on complex plane!

$$dw = \left(\frac{dw}{dz}\right) dz$$

↳ contains dilation factor $|dw/dz|$ + phase $\text{Arg}(dw/dz)$

but 

- This group is therefore ∞ dimensional since an analytic function is determined by an ∞ number of coefficients in its Laurent series about any point.

Q: Can we treat z and $\bar{z}$ as independent coordinates?

A: - Technically, no, but we would like to

What we do:

extend $(x^1, x^2) \in \mathbb{R}^2$ to $(x^1, x^2) \in \mathbb{C}^2$

- We then transform coordinates to $z, \bar{z}$ by a change of variables
- Can then treat $(z, \bar{z})$ as our independent coordinates on $\mathbb{C}^2$, with the caveat that the physical space is the submanifold $\bar{z} = z^*$ (real surface) on which we recover $(x, y) \in \mathbb{R}^2$
- Is useful because will see things naturally split up into $z, \bar{z}$ operators which will act on $\mathbb{C}^2$ naturally, and we can then impose $\bar{z} = z^*$ later!

3. Global Conformal Transformations:

- So far Eq'ns (7) are a local statement
- In order to form a group, the mappings should be invertible and map the reimann sphere $\mathbb{C} \cup \infty$ to itself
- What distinguishes?

In general we may write a Holomorphic CT generated by a vector field as:

$V(z) = -\sum_{n \in \mathbb{Z}} a_n l_n$ where l_n's are the generators

Study: (Local Properties)

Writing $ds^2 = dz\, d\bar{z} = dx^2 + dy^2$, then under an infinitesimal CT,

$$ds^2 \longrightarrow \frac{\partial f}{\partial z} \frac{\partial \bar{f}}{\partial \bar{z}} dz\, d\bar{z} \qquad (z \to f(z) \ldots)$$

$$f(z) = z - \varepsilon_n z^{n+1} \qquad \bar{f}(\bar{z}) = \bar{z} - \bar{\varepsilon}_n \bar{z}^{n+1} \qquad n \in \mathbb{Z}$$

then

$$\delta z = \varepsilon_n l_n \qquad \delta \bar{z} = \bar{\varepsilon}_n \bar{l}_n$$

with

$$\ell_n = -z^{n+1}\partial \quad , \quad \bar{\ell}_n = -\bar{z}^{n+1}\bar{\partial} \qquad \text{where } \partial \equiv \partial/\partial z$$

$n \in \mathbb{Z}$ $\qquad\qquad\qquad\qquad\qquad \bar{\partial} \equiv \partial/\partial \bar{z}$

then we find

$$[\ell_m, \ell_n] f(z) = z^{m+1}\partial(z^{n+1}\partial f) - z^{n+1}\partial(z^{m+1}\partial f)$$

$$= z^{m+1}(n+1)z^n \partial f + z^{m+1}z^{n+1}\partial^2 f$$

$$- z^{n+1}(m+1)z^m \partial f - z^{n+1}z^{m+1}\partial^2 f$$

$$= -z^{m+n+1}(m-n)\partial f = (m-n)\ell_{m+n} f$$

so find

$$\boxed{\begin{aligned} [\ell_m, \ell_n] &= (m-n)\ell_{m+n} \\ [\bar{\ell}_m, \bar{\ell}_n] &= (m-n)\bar{\ell}_{m+n} \\ [\ell_m, \bar{\ell}_n] &= 0 \end{aligned}} \qquad (8.)$$

- Just 2 copies of the Virasoro Algebra! (Classical)
- In quantum case, we will see them develop a central charge!
- Now can see that not all ℓ_n's (generators of local CT's) are defined on $\mathbb{C} \cup \infty$:

$$v(z) = -\sum a_n \ell_n = \sum_{n\in\mathbb{Z}} a_n z^{n+1}\partial$$

then

- Nonsingular $z \to 0 \Rightarrow a_n \neq 0$ for $n \geq -1$ only
- Nonsingular $z \to \infty$

  Perform $z \to (-1/\omega)$

$$v(z) = \sum a_n \left(\frac{-1}{\omega}\right)^{n+1}\left(\frac{\partial z}{\partial \omega}\right)^{-1} = \sum a_n \left(\frac{-1}{\omega}\right)^{n-1}\partial_\omega$$

so as $\omega \to 0$ $(z \to \infty) \Rightarrow a_n \neq 0$ for $n \leq 1$ only

$\Rightarrow$ Only CT generated by $a_n \ell_n$ for $n = 0, \pm 1$ are globally defined!

- This is the finite dimensional subgroup generated by
  $\{l_{-1}, l_0, l_1\} \cup \{\bar{l}_{-1}, \bar{l}_0, \bar{l}_1\}$
- These are the ones we saw before with no central charge term
  $([L_m, L_n] = (m-n)L_{m+n} + c/12\,(m^3-m)\,\delta_{m,-n})$

These transformations are just

$l_{-1}: z \to z - \epsilon \qquad \bar{l}_{-1}: \bar{z} \to \bar{z} - \epsilon$ <u>Translations</u>

$l_0: z \to z - \epsilon z \qquad \bar{l}_0: \bar{z} \to \bar{z} - \bar{\epsilon}\bar{z}$

or since $l_0 + \bar{l}_0 \sim \tau$ then $\tau \to \tau + \epsilon'$

so <u>Dilations</u> $\quad i(l_0 - \bar{l}_0) \sim \sigma$: <u>Rotations</u>

$l_0 = -z\partial_z$

while

$l_1 = -z^2 \partial_z \qquad$ SCT

- the generators $l_n + \bar{l}_n$ and $i(l_n - \bar{l}_n)$ preserve the real surface

Putting this together, we see that l_0, l_1, l_{-1} generate IT's:

$$\delta z = \alpha + \beta z + \gamma z^2$$

which are the generators of $SL(2,\mathbb{R})$

- The finite form is given by Möbius transformations

$$z \to z' = \frac{az+b}{cz+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$$

We can see this by expanding about $\qquad a,b,c,d \in \mathbb{R}$

$a = 1 = d \quad c = b = 0 \qquad ad - bc = 1$

$$\hookrightarrow z \to \frac{(1+\delta a)z + \delta b}{\delta c\, z + 1 + \delta d} = [(1+\delta a)z + \delta b](1 - \delta d - \delta c\, z)$$

$$\Rightarrow \delta z = \delta b + (\delta a - \delta c - \delta d)z - \delta c\, z^2$$

r

adding $\bar{l}_0, \bar{l}_1, \bar{l}_{-1}$ we generate $SL(2,\mathbb{C})/\mathbb{Z}_2$

$$z' = \frac{az+b}{cz+d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \qquad ad-bc=1$$

$a, b, c, d \in \mathbb{C}$

- These are the only globally defined invertible maps of $\mathbb{C} \cup \infty$ one-one onto itself,
- $/\mathbb{Z}_2$ because can change $a, b, c, d \to - \ldots$ gives same
- Note $SL(2,\mathbb{C}) \sim SO(3,1)$ so haven't really learned anything new!

In this language

$$\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} : \text{Translations} \qquad \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} : \text{Rotations}$$

$$\text{Dilations} : \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \qquad \text{Special} : \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}$$

with $B = a^1 + i a^2 \qquad C = b^1 - i b^2$

- 6 parameters ✓

§ Primary Fields

- Under CT's we saw that

$$ds^2 \to \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right) ds^2$$

- We define the basic objects of a CFT in 2 dimensions called primary fields that transform under CT's as:

$$\phi'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(z'(z), \bar{z}'(\bar{z}))$$

of weight $(h, \bar{h})$ so constructed to leave

$$\Phi(z,\bar{z})\, dz^h \, d\bar{z}^{\bar{h}} \quad \text{invariant!}$$

- This is the same as requiring them to transform as tensors

$$A_{\mu\ldots\nu}(x) \to \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \cdots \frac{\partial x'^{\beta}}{\partial x^{\nu}} A_{\alpha\ldots\beta}(x)$$

under CT's D_0

- Will take $h = \bar{h} \in \mathbb{Z}$ to avoid multi-valued fields
- Not all fields are primary, those not are called secondary

Under an ICT, a primary field transforms as:

$$z' = z + \varepsilon(z) \qquad \bar{z}' = \bar{z} + \bar{\varepsilon}(\bar{z})$$

so

$$(\partial f/\partial z)^{h} = 1 + h\partial\varepsilon + O(\varepsilon^2)$$

$$(\partial \bar{f}/\partial \bar{z})^{\bar{h}} = 1 + \bar{h}\bar{\partial}\bar{\varepsilon} + O(\bar{\varepsilon}^2)$$

$$\phi(z', \bar{z}') = 1 + \varepsilon\partial\phi + \bar{\varepsilon}\bar{\partial}\phi + O(\varepsilon, \bar{\varepsilon}^2)$$

and

$$\boxed{\begin{aligned}\phi'(z,\bar{z}) &= \phi(z,\bar{z}) + \delta_{\varepsilon,\bar{\varepsilon}}\,\phi(z,\bar{z}) \\ \delta_{\varepsilon,\bar{\varepsilon}} &= \left(h\partial\varepsilon + \varepsilon\partial + \bar{h}\bar{\partial}\bar{\varepsilon} + \bar{\varepsilon}\bar{\partial}\right)\end{aligned}} \qquad (9)$$

Free Fields + OPE's

- Typically correlation functions have singularities when the positions of 2 or more fields coincide, or

$$\phi_{av} \equiv \frac{1}{V}\int_V d^2x\, \phi(x)$$

of a field in a volume V has a variance $\langle \phi_{av}\phi_{av}\rangle$ that $\to \infty$ as $V \to 0$

$$\frac{(1+\delta a)z + \delta b}{\delta c z + 1 + \delta d}$$

$$= \frac{(1+\delta a)z + \delta b}{1 + (\delta d + \delta c z)} = \big((1+\delta a)z + \delta b\big)(1 - \delta d - \delta c z)$$

$$= (\delta a - \delta c - \delta d)z + \delta b - \delta c z^2$$

- The Operator Product Expansion is the representation of a product of operators at positions z and w by a sum of terms, each being a single operator well defined as $z \to w$ with coefficients (c number) of $(z-w)$ that may diverge as $z \to w$

  i.e.

$$A(z)B(w) = \sum_{n=-\infty}^{\infty} \frac{\{AB\}_n(w)}{(z-w)^n}$$

with the composite field $\{AB\}_n(w)$, nonsingular @ $w = z$

- This is understood as an operator eq'n $\emptyset$.

Ex:

Consider the free massless boson ϕ;

$$S = g/_2 \int d^2x \, \partial_\mu \phi \, \partial^\mu \phi$$

or equivilantly the free bosonic string with $\phi \to X^\mu$ $g \to T$

- We may then calculate the two-point correlation function or propagator

may rewrite S as:

$$S = 1/_2 \int d^2x \int d^2y \, \phi(x) A(x,y) \phi(y)$$

with

$$A(x,y) = g\,\delta(x-y)(-\partial^2)$$

then the propagator is just

$$K(x,y) = \langle \phi(x)\phi(y) \rangle = A^{-1}(x,y)$$

or $g(-\partial_x^2)\, K(x,y) = \delta(x-y)$ (10)

by Rotational invariance $K(x,y)$ should depend only on $r = |x-y|$

- Integrating (10), we find in polar coordinates

$$1 = 2\pi g \int_0^r \rho\, d\rho \left(-\tfrac{1}{\rho}\, \partial/\partial\rho (\rho K'(\rho)) \right)$$

$$= 2\pi g\, (-r K'(r))$$

$$\Rightarrow \quad K(r) = \frac{-1}{2\pi g} \ln r + \text{const}$$

or $$\langle \phi(x)\phi(y) \rangle = \frac{-1}{4\pi g} \ln (x-y)^2 + \text{const}$$

Going to complex coordinates, this is just

$$\langle \phi(z,\bar z)\, \phi(w,\bar w) \rangle = \frac{-1}{4\pi g} \left(\ln (z-w) + \ln (\bar z - \bar w) \right) + \text{const}$$

$$= \frac{-1}{4\pi g} \ln |z-w|^2 + \text{const}$$

Taking derivatives, then

$$\langle \partial_z \phi(z,\bar z)\, \partial_w \phi(w,\bar w) \rangle = \frac{-1}{4\pi g} \frac{1}{(z-w)^2}$$

$$\langle \partial_{\bar z} \phi(z,\bar z)\, \partial_{\bar w} \phi(w,\bar w) \rangle = \frac{-1}{4\pi g} \frac{1}{(\bar z-\bar w)^2}$$

So that we see that

$$\partial\phi(z)\, \partial\phi(w) = \frac{-1}{4\pi g} \frac{1}{(z-w)^2} + \text{finite}$$

and similar for barred.

- The energy-momentum tensor is

$$T_{\mu\nu} = g\left(\partial_\mu \phi\, \partial_\nu \phi - \tfrac{1}{2} \eta_{\mu\nu} \partial_\rho \phi\, \partial^\rho \phi \right)$$

which becomes (we will see)

$T(z) = -2\pi g :\partial\phi\,\partial\phi:$ in complex coordinates
Quantum version!

$: \; :$ = normal ordered

then by Wick's Theorem may write this as

$$T(z) = -2\pi g \lim_{w\to z} \left(\partial\phi(z)\,\partial\phi(w) - \langle \partial\phi(z)\,\partial\phi(w) \rangle \right)$$

Using this and Wick's Theorem the OPE of $T(z)$ with $\partial\phi$ can be calculated:

$$T(z)\,\partial\phi(w) = -2\pi g :\partial\phi(z)\,\partial\phi(z):\,\partial\phi(w)$$

$$= -4\pi g :\partial\phi(z)\,\partial\phi(z):\,\partial\phi(w) + \text{finite}$$

$$= \frac{\partial\phi(z)}{(z-w)^2} + \text{finite terms}$$

// Wick's Theorem

$T(\phi_1\phi_2 \cdots \phi_n) = :\phi_1 \cdots \phi_n: + :\text{all posible contractions}:$

where contraction $= \langle \quad \rangle$ (propagator)

ex: $T(\phi_1\phi_2) = :\phi_1\phi_2: + \langle \phi_1\phi_2 \rangle$

//

expanding $\partial\phi(z)$ around w, we get

$$T(z)\,\partial\phi(w) \sim \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial_w^2\phi(w)}{(z-w)} + \text{finite}$$

We may also compute

$$T(z)T(w) = 4\pi^2 g^2 :\partial\phi(z)\partial\phi(z)::\partial\phi(w)\partial\phi(w):$$

$$\sim \frac{1/2}{(z-w)^4} - 4\pi g \frac{:\partial\phi(z)\partial\phi(w):}{(z-w)^2}$$

2 double contractions

4 single contractions

again expandin $\partial\phi(z)$ around w $\rightarrow$

$$\boxed{T(z)T(w) \sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}} \qquad (11)$$

- we will come back to this!

Radial Quantization and the Operator Formalism

- First make contact with string theory, although this seems quite general

Consider the world sheet of a closed string (the cylinder)

- (If we had a general Euclidean theory just compactify one dimension)

parametrized by $\sigma \in [0, 2\pi]$ $\tau \in (-\infty, \infty)$

- We first make a wick rotation

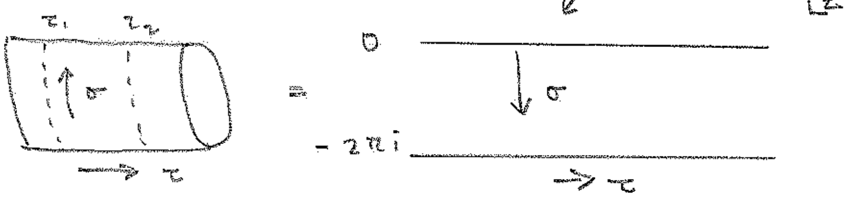
$\tau \to -i\tau$

or $\sigma^{\pm} = \tau \pm \sigma \to -i(\tau \pm i\sigma)$

to give us a Euclidean metric (not necessary if already in)

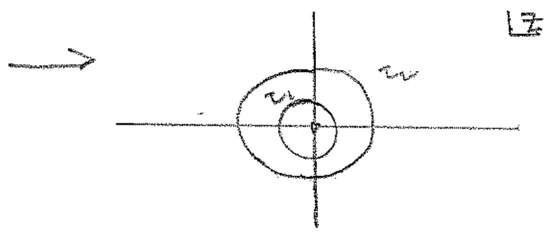
treated as $\in \mathbb{C}^2$ and make a change of variables to

$z' = \tau - i\sigma$ $\quad \bar{z}' = \tau + i\sigma$



- We then map the cylinder to the complex plane via the conformal transformation

$$\boxed{z = e^{z'} = e^{\tau - i\sigma} \qquad \bar{z} = e^{\bar{z}'} = e^{\tau + i\sigma}} \tag{12}$$



- This map will not change the theory if its conformally invariant!

Properties:

- $\tau = -\infty \to z = 0$, $\tau = \infty \to z = \infty$
- equal τ slices $\to$ circles of constant radius
- $\int d\sigma \to \oint_{C_0}$

- $T(\ ) \to$ Radial ordering
- $\sigma \to \sigma + \theta \Rightarrow z \to e^{-i\theta} z$ (Rotations)
- $\tau \to \tau + a \Rightarrow z \to e^{a} z$ (Dilations)

$\Rightarrow$ Generator of dilations will take role of Hamiltonian!

- Such a procedure is called Radial Quantization for reasons we shall soon see!
- Products of fields defined only in radial order
- This mapping is why we claimed $l_0 + \bar{l}_0 \sim \tau$ gave Dilations while $i(l_0 - \bar{l}_0) \sim \sigma$ gave Rotations
- For our primary fields, under Dilations $z \to \lambda z$, then

$$\phi \to \lambda^{h+\bar{h}} \phi$$

So $h+\bar{h}$ is called the scaling dimension while for Rotations,

$$z \to e^{-i\theta} z$$

$$\phi \to e^{-i(h-\bar{h})\theta} \phi$$

So $h-\bar{h}$ is called conformal spin

- say fields with h or $\bar{h} = 0$ are "chiral" since $z \sim \tau - i\sigma$, $\bar{z} \sim \tau + i\sigma$ this corresponds to only left or right moving strings!

- Considering the transformation
$$z = e^{z'} \quad \bar{z} = e^{\bar{z}'} \quad \text{or} \quad z' = \ln z \quad \bar{z}' = \ln \bar{z}$$
we find for primary fields:
$$\phi(z)_{plane} = \left(\frac{1}{z}\right)^{h} \phi(z')_{cylinder}$$

so that if $\phi(z')_{cylinder}$ has a mode expansion
$$\phi(z')_{cylinder} = \sum_{-\infty}^{\infty} \phi_n e^{-nz'} = \sum \phi_n z^{-n}$$

then on the plane this is just
$$\phi(z)_{plane} = \sum z^{-n-h} \phi_n$$
$$\Rightarrow$$
$$\phi_n = \oint_{C_0} \frac{dz}{2\pi i} \phi(z) z^{n+h-1} \quad , \; n+h \in \mathbb{Z}$$

b) Energy momentum Tensor
- Saw in 1st Chapter that $T_{\alpha\beta}$ was traceless
$$\boxed{T_{+-} = T_{-+} = 0}$$
and conserved
$$\partial^{\alpha} T_{\alpha\beta} = 0 \Rightarrow \partial^{+} T_{++} + \partial^{-} T_{-+} = 0$$
$$\partial^{-} T_{+-} + \partial^{-} T_{--} = 0$$
or $\boxed{\partial_{-} T_{++} = \partial_{+} T_{--} = 0}$

- We saw that in going to the plane
$\pm \to z, \bar{z}$
so that these conditions become
$$\boxed{\begin{aligned} T_{z\bar{z}} &= 0 \\ \partial T_{\bar{z}\bar{z}} = \bar{\partial} T_{zz} &= 0 \end{aligned}} \qquad (13)$$

This implies

$$\boxed{T_{zz} \equiv T(z) \qquad T_{\bar{z}\bar{z}} = \bar{T}(\bar{z})}$$

- So $T_{\alpha\beta}$ given by two non-vanishing functions that are analytic and anti analytic

- From 13, we find that if $T(z)$ is conserved, so is $\varepsilon(z)\,T(z)$ for any $\varepsilon = \varepsilon(z \text{ only})$
  ↳ This is the ∞ of conserved currents remarked earlier for $d=2$

- In usual FT's, a conserved charge $Q = \int d^{d-1}x\; j_0(x)$ integrated over a fixed time slice generates the infinitesimal symmetry variation
  $\delta_\varepsilon A = \varepsilon\,[Q, A]$

- Since now fixed time = constant radius this suggests we should take

$$\boxed{T_\varepsilon \equiv \oint_{C_0} \frac{dz}{2\pi i}\, \varepsilon(z)\, T(z)} \qquad (14)$$

which should generate ICT's $z \to z' = z + \varepsilon(z)$
i.e. we should expect

$$\delta_\varepsilon \phi(w) = [T_\varepsilon, \phi(w)] \qquad (15)$$

Then we should expect

$$\delta_\varepsilon \phi(\omega) = [T_\varepsilon, \phi(\omega)] \quad !$$

- Including the barred part, then

$$\boxed{Q = \frac{1}{2\pi i}\oint \left(dz\, \varepsilon(z) T(z) + d\bar{z}\, \bar{T}(\bar{z}) \bar{\varepsilon}(\bar{z}) \right) = Q_\varepsilon + Q_{\bar{\varepsilon}}} \quad (16)$$

- Remembering that a product should be radial ordered, then must have

$$\delta_\varepsilon \phi(\omega) = \oint_{\substack{C_0 \\ |z|>|\omega|}} \frac{dz}{2\pi i} \varepsilon(z) T(z) \phi(\omega) - \oint_{\substack{C_0 \\ |z|<|\omega|}} \frac{dz}{2\pi i} \varepsilon(z) T(z) \phi(\omega)$$

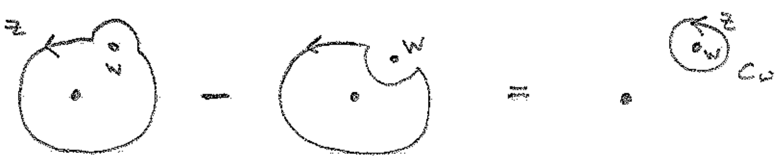
R understood from now on

or

$$= \frac{1}{2\pi i} \left(\oint_{|z|>|\omega|} - \oint_{|z|<|\omega|} \right) \left(dz\, \varepsilon(z) R\left(T(z)\phi(\omega)\right)\right)$$

$$R\left(A(z)B(\omega)\right) = \begin{cases} A(z)B(\omega) & |z|>|\omega| \\ B(\omega)A(z) & |z|<|\omega| \end{cases}$$

These contours are just:



so that

$$\boxed{\delta_\varepsilon \phi(\omega) = \oint_{C_\omega} \frac{dz}{2\pi i} \varepsilon(z) T(z) \phi(\omega)} \quad (17)$$

Now from the Def'n

$$\phi(z,\bar{z}) \to \phi'(z,\bar{z}) = \left(\frac{\partial z'}{\partial z}\right)^h \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{\bar{h}} \phi(z'(z), \bar{z}'(\bar{z}))$$

with $\bar{h} = 0$ for which we found:

$$\delta_\varepsilon \phi(z) = \left(h\, \partial\varepsilon(z) + \varepsilon(z)\partial \right) \phi(z)$$

Then using

$$\oint_{C_w} \frac{dz}{(2\pi i)} \frac{f(z)}{(z-w)^n} = \frac{1}{(n-1)!} f^{(n-1)}(w)$$

Then if

$$T(z)\,\phi(w) = \frac{h\,\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{(z-w)} + \text{finite}$$

$$\delta_\varepsilon \phi(w) = \oint_{C_w} \frac{dz}{2\pi i} \left\{ h\frac{\phi(w)\varepsilon(z)}{(z-w)^2} + \frac{\partial\phi(w)}{(z-w)}\varepsilon(z) + \dots \right\}$$

$$= \left(h\,\partial\varepsilon(w) + \varepsilon(w)\partial\right)\phi(w) \quad \checkmark$$

Doing the same for the barred:

$$\boxed{\begin{aligned} T(z)\,\phi(w) &= \frac{h\,\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{(z-w)} + \dots \\ \bar{T}(\bar{z})\,\phi(\bar{w}) &= \frac{\bar{h}\,\phi(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\,\phi(\bar{w})}{(\bar{z}-\bar{w})} + \dots \end{aligned}} \quad (18)$$

- So requiring that Q induce these transformations has given us the OPE!
- We can take this as the definition of a conformal field of weight $(h, \bar{h})$!
- Looking at (11) we could guess that

$$\boxed{T(z)\,T(w) = \frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{1}{(z-w)}\partial T(w) + \dots} \quad (19)$$

is the OPE of T with itself, where c is called the central charge.

- Saw for free boson $c=1$

- Can also check by considering the properties of 2 conformal infinitesimal transformations

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] = \delta_{(\varepsilon_1 \partial \varepsilon_2 - \varepsilon_2 \partial \varepsilon_1)}$$

and the definitions of δ_ε in terms of T as we did for $T(z)\phi(w)$!

- Let us pause to make contact with String theory
  - Saw already $T_{zz} = T(z)$, $T_{\bar{z}\bar{z}} = \bar{T}(\bar{z})$
    $T_{z\bar{z}} = 0$

  taking $l_s = \sqrt{2\alpha'} = 1$ the coordinate expansions for X^μ become:

$$\boxed{\begin{aligned} X^\mu_R(\tau,\sigma) &\to X^\mu_R(z) = 1/2\, x^\mu - i/4\, p^\mu \ln z + \tfrac{i}{2}\sum \tfrac{1}{n}\alpha_n^\mu z^{-n} \quad (20)\\ X^\mu_L(\tau,\sigma) &\to X^\mu_L(\bar{z}) = \tfrac{1}{2}x^\mu - i/4\, p^\mu \ln \bar{z} + \tfrac{i}{2}\sum \tfrac{1}{n}\tilde{\alpha}_n^\mu \bar{z}^{-n} \quad (21)\end{aligned}}$$

and

$$\boxed{\begin{aligned} \partial X^\mu(z,\bar{z}) &= -\frac{i}{2}\sum_{-\infty}^{\infty} \alpha_n^\mu z^{-n-1} \\ \bar{\partial} X^\mu(z,\bar{z}) &= -\frac{i}{2}\sum_{-\infty}^{\infty} \tilde{\alpha}_n^\mu \bar{z}^{-n-1}\end{aligned}} \quad (22)$$

which gives

$$\boxed{\begin{aligned} T_X(z) &= -2 :\partial X \cdot \partial X: = \sum_{-\infty}^{\infty} \frac{L_n}{z^{n+2}} \quad (23)\\ \bar{T}_X(\bar{z}) &= -2 :\bar{\partial} X \cdot \bar{\partial} X: = \sum_{-\infty}^{\infty} \frac{\bar{L}_n}{\bar{z}^{n+2}} \quad (24)\end{aligned}}$$

X; constructed from X^μ coordinates

- However, this last expression is quite general

  see References

  these can be inverted to give

$$\boxed{L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z) \qquad \bar{L}_n = \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{n+1} \bar{T}(\bar{z})} \qquad (25)$$

- We can calculate the propagator for X^μ's by analogy with free boson ex, giving

$$X^\mu(z)\, X^\mu(w) = -\frac{1}{4}\eta^{\mu\nu} \ln(z-w) + \ldots$$

So its OPE is not meromorphic! (not made of simple poles)

While looking at

$$\langle \partial X^\mu(z)\, \partial X^\nu(w) \rangle = -\frac{1}{4} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle 0 | \alpha_m^\mu \alpha_n^\mu | 0 \rangle z^{-m-1} z^{-n-1}$$

$$= -1/4 \sum_{m,n=1}^{\infty} \langle 0 | \alpha_m^\mu \alpha_{-n}^\mu | 0 \rangle z^{-m-1} w^{n-1}$$

$$= -\frac{\eta_{\mu\nu}}{4} \sum_{m,n=1}^{\infty} m\, \delta_{m,n} z^{-m-1} w^{n-1}$$

$$= -\frac{\eta^{\mu\nu}}{4} \sum_{n=1}^{\infty} n \left(\frac{w^n}{w}\right)\left(\frac{1}{z^{n+1}}\right) = -\frac{\eta^{\mu\nu}}{4} \sum_1^{\infty} n \left(\frac{w}{z}\right)^n \frac{1}{wz}$$

which for $|w| < |z|$ gives

$$= \frac{-\eta^{\mu\nu}}{4(z-w)^2}$$

So that

$$:\partial X^\mu(z)\, \partial X^\nu(z): = \lim_{w \to z} \left(\partial_z X^\mu(z)\, \partial_w X^\nu(w) + \frac{\eta^{\mu\nu}}{4(z-w)^2} \right)$$

so its a conformal field of $h=1$

and if we compute

$$T(z)\, \partial X(w) = \frac{\partial X(w)}{(z-w)^2} + \frac{\partial(\partial X(w))}{(z-w)} + \ldots$$

ν

- each of these will give a contribution 1 to the conformal anomaly so that in D dimensions $c = \Sigma = D$

From OPE (19), we see that
$T(z)$ itself is not a conformal field
unless $c = 0$!
(meaning?)
If $c=0$, then it has dimension $(2,0)$
and $\bar{T}(\bar{z}) \sim (0,2)$

- We now look at how the energy momentum tensor transforms under finite CT's

$z \to w(z)$:

$$\boxed{\begin{aligned}(\partial w)^2 T'(w) &= T(z) - \frac{c}{12} S(w,z) \\ S(w,z) &\equiv \frac{2(\partial w)(\partial^3 w) - 3(\partial^2 w)^2}{2(\partial w)^2}\end{aligned}} \quad (26)$$

→ Schwarzian derivative

- This "exponentiation" is not obvious, but we can check it gives the correct result for an IT
- Under an IT

$w(z) = z + \varepsilon(z)$

$$S(w,z) = \frac{\partial^3 w}{\partial w} - \frac{3}{2}\left(\frac{\partial^2 w}{\partial w}\right)^2$$

$$= \frac{\partial^3 \varepsilon}{1+\partial\varepsilon} - \frac{3}{2}\left(\frac{\partial^2 \varepsilon}{1+\partial\varepsilon}\right)^2 \approx \partial^3 \varepsilon$$

So

$$T'(w) = (\partial w)^{-2}\left(T(z) - \frac{c}{12}\partial^3\varepsilon\right)$$

$$T'(z+\varepsilon) = (1-2\partial\varepsilon)\left(T(z) - \frac{c}{12}\partial^3\varepsilon\right)$$

or

$$\delta_\varepsilon T(z) = -\frac{1}{12}\partial^3\varepsilon - 2(\partial\varepsilon)T(z) - \varepsilon\partial T(z)$$

while (19) gives

$$\delta_\varepsilon T(w) = \oint \frac{dz}{2\pi i} \varepsilon(z) [T(z), T(w)] = \oint_w \frac{dz}{2\pi i} \varepsilon(z) T(z) T(w)$$

$$= \oint \frac{dz}{2\pi i} \varepsilon(z) \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots \right)$$

$$= 2\,\partial\varepsilon(w) T(w) + \varepsilon(w)\,\partial T(w) + \frac{c}{12} \partial^3 \varepsilon(w)$$

But

$\delta_\varepsilon T(w) = -\delta_\varepsilon T(z)$ since $z \sim w - \varepsilon$ so they agree!

Also, (26) satisfy the group property by considering two successive CT's,

$w(u(z))$, $\partial = \partial/\partial z$

then

$$(\partial w)^2 T(w) = T(z) - \frac{c}{12} S(u,z) - \frac{c}{12} (\partial u)^2 S(w,u)$$

Then since $S(w,z)$ satisfies

$$S(w,z) = S(u,z) + (\partial u)^2 S(w,u)$$

which can be shown by

$$\frac{dw}{du} = \left(\frac{du}{dz}\right)^{-1} \frac{dw}{dz} = \frac{w'}{u'}$$

and

$$\frac{d^2 w}{du^2} = \frac{w'' u' - w' u''}{(u')^3}$$

$$\frac{d^3 w}{du^3} = \frac{w'''(u')^2 - 3w'' u'' u' - w' u''' u' + 3w'(u'')^2}{(u')^5}$$

into $S(w,u)$

- can also show that

  $S(w,N)$, $w = \dfrac{az+b}{cz+d}$ $(ad - bc = 1)$

  $= 0$

- Can show (?) it is the only possible addition to the tensor transformation that satisfies the group properties

Finally, using (23)

$$T(z) = \sum_{-\infty}^{\infty} \frac{L_n}{z^{n+2}} \quad , \quad L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z)$$

we can see that the QM generators get the appropriate central charge:

$$[L_m, L_n] = \frac{1}{(2\pi i)^2} \oint dw\, w^{n+1} \oint_w dz\, z^{m+1}\, T(z) T(w)$$

$$= \oint \frac{dw}{2\pi i} w^{n+1} \oint \frac{dz}{2\pi i} z^{m+1} \left(\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{(z-w)} \partial T(w) + \dots \right)$$

$$= \oint \frac{dw}{2\pi i} \left\{ \frac{c}{12} (m^3 - m)\, w^{m+n-1} + 2(m+1)\, w^{n+m+1}\, T(w) \right.$$

$$\left. + w^{m+n+2} \partial T(w) \right\}$$

$$= \frac{c}{12} (m^3 - m)\, \delta_{m+n,0} + 2(m+1) L_{m+n}$$

$$- \oint \frac{dw}{2\pi i} (m+n+2)\, w^{m+n+1}\, T(w)$$

IBP's

$$= c/12\, (m^3 - m)\, \delta_{m+n,0} + (m-n) L_{m+n}$$

so

$$\boxed{[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} (m^3 - m)\, \delta_{m+n,0}} \qquad (27)$$

Similarly

$$\boxed{\begin{aligned} [\bar{L}_m, \bar{L}_n] &= (m-n) \bar{L}_{m+n} + \frac{c}{12} (m^3 - m)\, \delta_{m+n,0} \\ [L_n, \bar{L}_m] &= 0 \end{aligned}}$$

OPE $T(z) \bar{T}(\bar{w}) \sim 0$

3) Hilbert space + Operators

- We assume $\exists$ some vacuum state $|0\rangle$ upon which a Hilbert Space is constructed by application of creation operators or the like.
- For an interacting field, we assume that $\mathcal{H}$ is the same as for the free field, except that the energy eigenstates are different, and that the interaction "turns on/off" so that as $t \to \pm\infty$ we can identify assymptotic states:

$$\phi_{in} \sim \lim_{t \to -\infty} \phi(x,t) \quad \text{is free}$$

- In radial Quantization, this assymptotic field reduces to a single operator:

$$|\phi_{in}\rangle = \lim_{z,\bar{z} \to 0} \phi(z,\bar{z})|0\rangle$$

(Remember, $z \to 0$ is $t \to -\infty$)

Hermitian Product:

- must introduce a bilinear product to have a valid $\mathcal{H}$!
- We do this indirectly by defining an "out" state and the action of Hermition conjugation
- In Euclidean space, since $\tau = it$ then $\tau \to -\tau$ upon hermitian conjugation if t is left unchanged

In radial Quantization, this is

$$z \to 1/z^*,$$

with this in mind, we define on real surface $\bar{z} = z^*$

$$\boxed{[\phi(z,\bar{z})]^\dagger = \bar{z}^{-2h} z^{-2\bar{h}} \, \phi(1/\bar{z}, 1/z)} \qquad (28)$$

for a field $(h,\bar{h})$ so that an assymptotic out state;

$$\langle \phi_{out}| \equiv |\phi_{in}\rangle^{\dagger}$$

has a well defined inner product with $|\phi_{in}\rangle$:

$$\langle \phi_{out}|\phi_{in}\rangle = \lim_{z,\bar{z},w,\bar{w}\to 0} \langle 0|\phi(z,\bar{z})^{\dagger}\phi(w,\bar{w})|0\rangle$$

$$= \lim_{z,\bar{z},w,\bar{w}\to 0} \bar{z}^{-2h} z^{-2\bar{h}} \langle 0|\phi(1/\bar{z}, 1/z)\phi(w,\bar{w})|0\rangle$$

$$= \lim_{\zeta,\bar{\zeta}\to\infty} \bar{\zeta}^{2h}\zeta^{2\bar{h}} \langle 0|\phi(\bar{\zeta},\zeta)\phi(0,0)|0\rangle$$

↑ Time ordered

However correlation functions of primaries

$$\langle \phi_1(w_1,\bar{w}_1)\dots\phi_n(w_n,\bar{w}_n)\rangle$$

$$= \prod_{i=1}^{n}\left(\frac{\partial w}{\partial z}\right)^{-h_i}_{w=w_i}\left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{h}_i}_{\bar{w}=\bar{w}_i}\langle \phi(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\rangle$$

fixes the 2 and 3 point functions

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle = \frac{C_{12}}{(z_1-z_2)^{2h}(\bar{z}_1-\bar{z}_2)^{2\bar{h}}} \quad \text{if} \begin{cases} h_1=h_2=h \\ \bar{h}_1=\bar{h}_2=\bar{h}\end{cases}$$

so that

$\langle \phi_{out}|\phi_{in}\rangle$ is independent of ζ and so is well defined

Mode Expansions:

- Can expand a conformal field of $(h,\bar{h})$ as we saw:

$$\phi(z,\bar{z}) = \sum_m\sum_n z^{-m-h}\bar{z}^{-n-\bar{h}}\phi_{m,n}$$

$$\phi_{m,n} = \frac{1}{2\pi i}\oint dz\, z^{m+h-1}\frac{1}{2\pi i}\oint d\bar{z}\,\bar{z}^{n+\bar{h}-1}\phi(z,\bar{z})$$

then

$$\phi(z,\bar{z})^{\dagger} = \sum\sum \bar{z}^{-m-h}z^{-n-\bar{h}}\phi^{\dagger}_{m,n}$$

$$= \bar{z}^{-2h} z^{-2h} \phi(1/\bar{z}, 1/z) = \bar{z}^{-2h} z^{-2\bar{h}} \sum\sum \phi_{m,n} \bar{z}^{m+h} z^{n+h}$$

$$= \sum\sum \phi_{-m,-n} \bar{z}^{-m-h} z^{-n-h}$$

so that

$$\boxed{\phi^{+}_{m,n} = \phi_{-m,-n}}$$

- This justifies our definition of expansion
- To be well defined, (the "in" and "out" states) the vacuum should satisfy

$$\boxed{\phi_{m,n} |0\rangle = 0 \qquad (m > -h, \; n > -h)} \qquad (29)$$

Dropping $\bar{z}$ dependance, we thus have

$$\boxed{\begin{aligned} \phi(z) &= \sum_{-\infty}^{\infty} z^{-m-h} \phi_m \\ \phi_m &= \frac{1}{2\pi i} \oint dz \, z^{m+h-1} \phi(z) \end{aligned}} \qquad (30)$$

§ <u>Hilbert Space of CFT's</u> (Rough)

- The vacuum state $|0\rangle$ should be invariant under global conformal transformations!

$\Rightarrow$ should be annihilated by L_{-1}, L_0 and L_1 (and $\bar{L}_{-1}$, $\bar{L}_0$, $\bar{L}_1$)

Remembering our expansions

$T(z) = \sum z^{-n-2} L_n \qquad \bar{T}(\bar{z}) = \ldots$

then

for $T(z)|0\rangle$ and $\bar{T}(\bar{z})|0\rangle$ to be well defined as $z, \bar{z} \to 0$, must have

$$\boxed{\begin{aligned} L_n |0\rangle &= 0 \\ \bar{L}_n |0\rangle &= 0 \end{aligned} \qquad n \geq -1} \qquad (31)$$

This also $\Rightarrow$ $\langle 0|T(z)|0\rangle = \langle 0|\bar{T}(\bar{z})|0\rangle = 0$
by condition $L^{+}_{m} = L_{-m}$ ☺

- We have defined the vacuum such that when primary fields act on it, they create assymptotic states, which are eigenstates of the Hamiltonian $\sim L_0 + \bar{L}_0$!

Can see this by considering

$$[L_n, \phi(w,\bar{w})] = \frac{1}{2\pi i}\oint dz\, z^{n+1} T(z)\,\phi(w,\bar{w})$$

$$= \frac{1}{2\pi i}\oint dz\, z^{n+1}\left\{ \frac{h\,\phi(w,\bar{w})}{(z-w)^2} + \frac{\partial\phi(w,\bar{w})}{(z-w)} + \dots \right\}$$

$$= h(n+1)\,w^n\phi(w,\bar{w}) + w^{n+1}\partial\phi(w,\bar{w}) \quad \begin{cases} (n \geq -1) \\ 0 \quad n \text{ other} \end{cases}$$

similarly

$$[\bar{L}_n, \phi(w,\bar{w})] = \bar{h}(n+1)\bar{w}^n(\phi(w,\bar{w})) + \bar{w}^{n+1}\bar{\partial}\phi(w,\bar{w})$$

Applying these to the assymptotic state

$$\boxed{|h,\bar{h}\rangle \equiv \phi(0,0)|0\rangle} \qquad (32)$$

then

$$L_0|h,\bar{h}\rangle = L_0\phi(0,0)|0\rangle = [L_0, \phi(0,0)]|0\rangle$$

$$\boxed{L_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle \quad \text{and} \quad \bar{L}_0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle} \qquad (33)$$

so $|h,\bar{h}\rangle$ is an eigenstate of H

while

$$\boxed{\begin{array}{l} L_n|h,\bar{h}\rangle = 0 \\ \bar{L}_n|h,\bar{h}\rangle = 0 \end{array}} \quad n > 0 \qquad (34)$$

These states are called the highest weight states!

- Excited states above $|h,\bar{h}\rangle$ may be obtained by applying ladder operators as such:

expand

$$\phi_n = \frac{1}{2\pi i}\oint dz\, z^{m+h-1}\,\phi(z)$$

and $[L_n, \phi]$, to get

$$[L_n, \phi_m] = \oint \frac{dw}{2\pi i}\, w^{h+m-1}\left(h(n+1)w^n\phi(w) + w^{n+1}\partial\phi(w)\right)$$

$$= \oint \frac{dw}{2\pi i}\, w^{n+m+h-1}\left(h(n+1) - (h+m+n)\right)\phi(w)$$

$$[L_n, \phi_m] = (n(h-1) - m)\,\phi_{m+n}$$

So that

$$[L_0, \phi_m] = -m\,\phi_m \qquad (35)$$

$\Rightarrow$ ϕ_m act as raising + lowering operators for eigenstates of L_0: each application of ϕ_{-m} $(m>0)$ increases the conformal dimension by m:

$$L_0(\phi_{-m}|h,\bar{h}\rangle) = ([L_0, \phi_{-m}] + \phi_{-m}L_0)|h,\bar{h}\rangle$$

$$= (m+h)\,\phi_{-m}|h,\bar{h}\rangle \quad ✓$$

The Generators L_{-m} $(m>0)$ also increase the conformal dimension since $[L_0, L_{-m}] = m L_{-m}$

$\Rightarrow$ excited states can be created:

$$L_{-k_1} L_{-k_2} \ldots L_{-k_n}|h\rangle \qquad (1 \leq k_1 \leq \ldots k_n)$$

these states are called <u>descendants</u> of $|h\rangle$

such a state is an eigenstate of $|h\rangle$ with eigenvalue

$$h' = h + k_1 + k_2 + k_3 + \dots k_n \equiv h + N$$

and

N is called the <u>level</u> of the descendant

- Important because of conformal properties: A CT on a state is obtained by acting on it with a suitable L_m. The subset of $\mathcal{H}$ generated by $|h\rangle$ and its descendants is closed under the action of generators and thus forms a representation or <u>module</u> of the Virasoro algebra.

  This subspace is called a <u>Verma module</u>

- We have already seen this!

  Physical states of the open string satisfy

$$L_0 |\phi\rangle = (1) |\phi\rangle$$

$$L_n |\phi\rangle = 0 \qquad n > 0$$

  So these are highest states with $h = 1$.

- Physical states for Bosonic String

  $|h\rangle$ with $h = \bar{h} = 1$ D_0