HET Journal club June 16TH 2009

Renormalisation

Note Title 6/16/2009

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- This subject is very vast.
- To try to compress it in only one talk is nearly impossible
- I won't go into much details, but rather I will try to explain in more physical terms with simple examples

$$\boxed{\text{In many theories of nature, i.e. theories that have been experimentally verified, the calculation of physical quantities initially gives } \infty.}$$

This is the statement of today's discussion

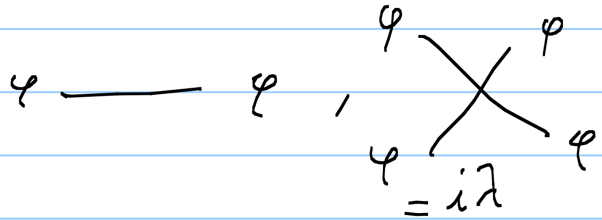
Examples: QED, Yang-Mills (QCD, Electro Weak)

Still, people managed to "understand" those infinities and make extraordinarily accurate predictions. (QED → magnetic moment of the electron initially, and many of collider experiments since decades ago)

How does it work?

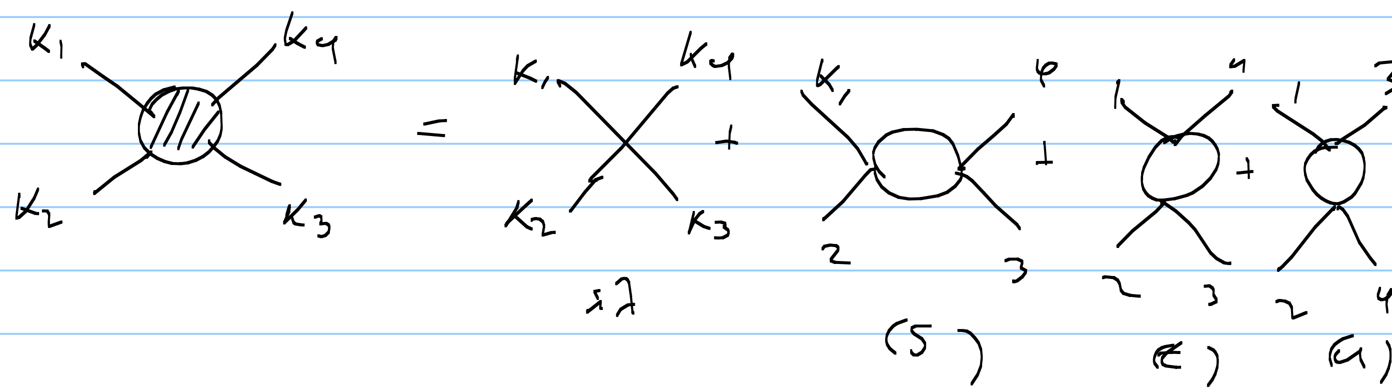
Let's begin with a toy theory

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4!}\varphi^4$$



: building blocks for interactions

Let's compute the amplitude of scattering of 4 of these scalar particles:

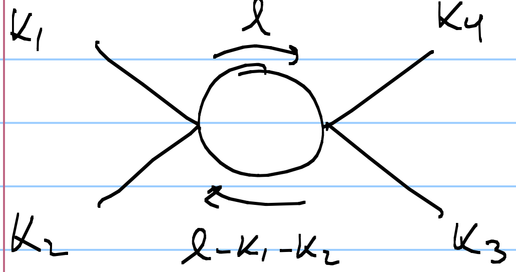


$+ \ldots\ldots$

This is a graphical way of writing

$$\mathcal{M} = \underbrace{i\lambda}_{\text{tree}} + \underbrace{O(\lambda^2)}_{\text{all the rest}}$$

Let's compute the first term of the rest



$$= \frac{1}{2}(i\lambda)^2 \underbrace{\int \frac{d^4 l}{(2\pi)^4} \frac{-i}{l^2-m^2+i\epsilon} \times \frac{-i}{(l-k_1-k_2)^2-m^2+i\epsilon}}_{\text{this gives } \infty}$$

We can actually see that it diverges due to high values of l. For large l

$$\sim \int dl \, l^3 \frac{1}{l^4} = \int^{\Lambda} \frac{dl}{l} = \log \Lambda \quad (\Lambda \to \infty)$$

We have a problem right away: the second term in this series expansion should be smaller compared to the first, but it gives ∞!!

How to make sense of this?

The modern point of view is: We should think of this theory a low energy effective theory of a more complete one.

From this point of view, it makes perfect sense that the cutoff Λ is really present in the theory.

OK!, but now the question is: What value of Λ should I use? It seems that the scattering amplitude (i.e: cross sections, and decay rates, etc) directly depend on it.

To answer this, let's compute the integrals:

$$M = -i\lambda + iC\lambda^2\left[\log\left(\frac{\Lambda^2}{(k_1+k_2)^2}\right) + \log\left(\frac{\Lambda^2}{(k_2+k_3)^2}\right) + \log\left(\frac{\Lambda^2}{(k_2+k_4)^2}\right)\right]$$
$$+ \mathcal{O}(\lambda^3)$$

or more concisely

$s \equiv (k_1+k_2)^2$
$t = (k_2+k_3)^2$
$u = (k_2+k_4)^2$

C: just a numerical constant

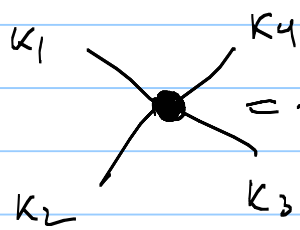
$$M = -i\lambda + iC\lambda^2\Bigg[\underbrace{\log\left(\frac{\Lambda^2}{s}\right) + \log\left(\frac{\Lambda^2}{t}\right) + \log\left(\frac{\Lambda^2}{u}\right)}_{\equiv L}\Bigg]$$
$$+ \mathcal{O}(\lambda^3)$$

λ: coupling constant
Λ: cutoff

Key point: λ is NOT the real physical coupling measured in experiments (QED experiments measure $\alpha = \frac{e^2}{4\pi}$)

But rather λ and Λ are functions of each other.

Experimentally, what one measures is an effective or physical coupling constant λ.



$$= -i\lambda_{PHYS} = \mathcal{M}$$

$\hookrightarrow$ full amplitude (nature doesn't care about our diagrammatic expansion)

Imagine we go and measure λ_{PHYS} by doing and experiment at a certain CM energy, i.e:

$$(k_1^0 + k_2^0)^2 = s_0$$
$$(k_2^0 + k_3^0)^2 = t_0$$
$$(k_2^0 + k_4^0)^2 = u_0$$

* Thus, our calculation at this energy should give this value of $\mathcal{M} = -i\lambda_{PHYS}$

$$-i\lambda_{PHYS} = -i\lambda + i C \lambda^2 \left[\log\left(\frac{\Lambda^2}{s_0}\right) + \log\left(\frac{\Lambda^2}{t_0}\right) + \log\left(\frac{\Lambda^2}{u_0}\right)\right] + O(\lambda^3)$$

or $-i\lambda_{PHYS} = -i\lambda + i C \lambda^2 L_0 + O(\lambda^3)$

Solving for λ: $\quad i\lambda = i\lambda_{PHYS} + i C \lambda^2 L_0 + O(\lambda^3)$

$\uparrow$ plug here recursively

$$i\lambda = i\lambda_{PHYS} + iC\left[\lambda_{PHYS} + C\lambda^2 L_0\right]^2 L_0 + O(\lambda^3)$$

$$i\lambda = i\lambda_{PHYS} + iC\lambda_{PHYS}^2 L_0 + O(\lambda^3)$$

↳ now we have λ in terms of experimentally measured quantities up to the order specified.

thus

$$M = -i\lambda + iC\lambda^2 L + O(\lambda^3) \quad \text{becomes}$$

$$M = -i\lambda_{PHYS} - iC\lambda_{PHYS}^2 L_0 + O(\lambda^3)$$
$$+ iC\left[i\lambda_{PHYS} + iC\lambda_{PHYS}^2 L_0 + O(\lambda^3)\right]^2 L + O(\lambda^3)$$

$$= -i\lambda_{PHYS} - iC\lambda_{PHYS}^2 L_0 + iC\left[i^2\lambda_{PHYS}^2 + O(\lambda_{PHYS}^3)\right]L$$
$$+ O(\lambda^3)$$

$$M = -i\lambda_{PHYS} + iC\lambda_{PHYS}^2 \underbrace{(L - L_0)} + O(\lambda_{PHYS}^3)$$

$$\Longrightarrow L - L_0 = \log\left(\frac{\cancel{\Lambda^2}}{s}\right) + \log\left(\frac{\cancel{\Lambda^2}}{t}\right) + \log\left(\frac{\cancel{\Lambda^2}}{u}\right)$$
$$- \log\left(\frac{\cancel{\Lambda^2}}{s_0}\right) - \log\left(\frac{\cancel{\Lambda^2}}{t_0}\right) - \log\left(\frac{\cancel{\Lambda^2}}{u_0}\right)$$

$$L - L_0 = \log\left(\frac{s_0}{s}\right) + \log\left(\frac{t_0}{t}\right) + \log\left(\frac{u_0}{u}\right)$$

Finally

$$\boxed{\mathcal{M} = -i\lambda_{PHYS} + iC\lambda_{PHYS}^2\left[\log\left(\frac{s_0}{s}\right) + \log\left(\frac{t_0}{t}\right) + \log\left(\frac{u_0}{u}\right)\right] + \mathcal{O}(\lambda_{PHYS}^3)}$$

This is at the heart of Renormalization

The dependence on the cutoff Λ has completely disappear when we express the amplitude in terms of physically observable quantities

This is what books sometimes call <u>the Miracle of renormalization</u>.

This "miracle" does not always happen; it happens for theories that are called Renormalizable.

The Beta function

Our calculation was

$$M = -i\lambda + i\lambda^2 C\left[\log\frac{\Lambda^2}{s} + \log\frac{\Lambda^2}{t} + \log\frac{\Lambda^2}{u}\right] + \mathcal{O}(\lambda^3)$$

The idea of renormalization is that λ and Λ balance each other out in order to yield M intact under a change of them

$$\delta M = -i\,\delta\lambda + 2i\lambda\,\delta\lambda\, C\left[\log\frac{\Lambda^2}{s} + \cdots\right]$$

$$+ i\lambda^2 C\left[\frac{2}{\Lambda} \times 3\,\delta\Lambda\right] + \mathcal{O}(\lambda^3)$$

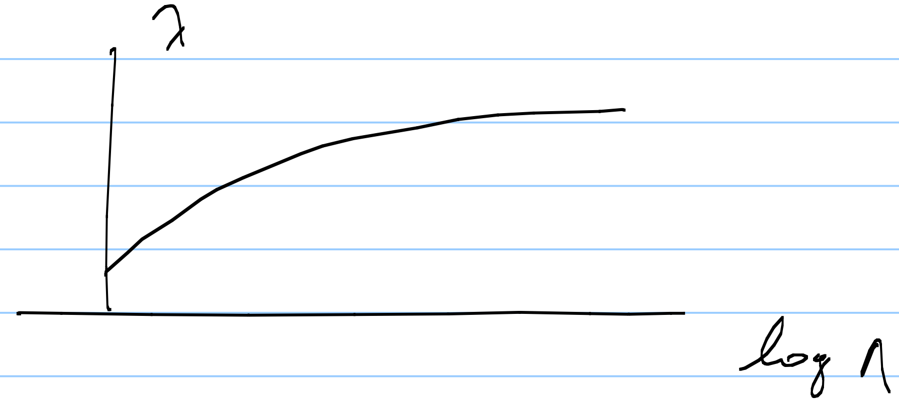
$$0 = \delta\lambda\left(-\cancel{i} + 2\cancel{i}\lambda C\overbrace{\left[\log\frac{\Lambda^2}{s} + \cdots\right]}^{L}\right) + 6\cancel{i}\lambda^2 C\,\frac{\delta\Lambda}{\Lambda}$$

$$0 = \delta\lambda\left(-1 + 2\lambda C L\right) + \frac{6\lambda^2 C}{\Lambda}\,\delta\Lambda$$

$$\Lambda\frac{\delta\lambda}{\delta\Lambda} = 6\lambda^2 C + \mathcal{O}(\lambda^3)$$

$$\beta \equiv \frac{d\lambda(\Lambda)}{d\ln\Lambda} = \frac{d\lambda}{d\Lambda}\frac{d\Lambda}{d\ln\Lambda} = \frac{d\lambda}{d\Lambda}\Lambda$$

$$\therefore \quad \beta(\lambda) = \underbrace{6\lambda^2 C}_{\text{positive}} + O(\lambda^3)$$



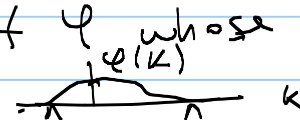
The coupling increases with the energy at which we put the cutoff.

— There's of course a more systematic and formal way of computing $\beta(\lambda)$.

Remark: This 4-point function in the path integral formulation corresponds to the calculation of

$$\int D\varphi\, \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\, e^{i\int d^4x\left[-\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4\right]}$$

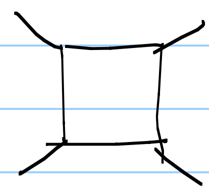
and restrict $\int D\varphi$ only to those values of φ whose Fourier Transform vanishes for $K \geq \Lambda$



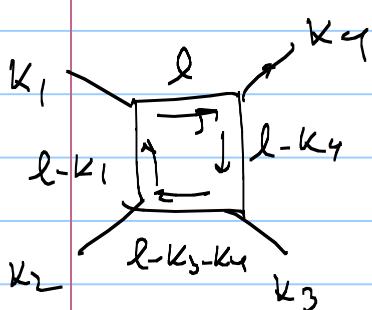
(2) Which theories are renormalizable and which are not?

- Infinite expressions come from loop integrals (that's why non-renormalizable theories still give good predictions at the tree level, even though they blow up at the one loop level)

Consider the diagram



in a φ^3 theory:



E : # of external legs
I : # of internal lines
L : # of closed loops
V_n : # of vertices that connect n lines.

$$\sim \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2+m^2} \frac{1}{(l-k_4)^2+m^2} \frac{1}{(l-k_3-k_4)^2+m^2} \frac{1}{(l-k_1)^2+m^2}$$

$L=1 \quad V_3=4$ # of l's in the num $= dL$
$I=4 \quad V_4=0$ # " " " dem $= 2I$
$E=4 \quad V_5=0$

$dL - 2I \equiv D$ (superficial degree of divergence)

$$\int^{\Lambda} dl \; l^{D-1}$$

if $D-1 = -1$

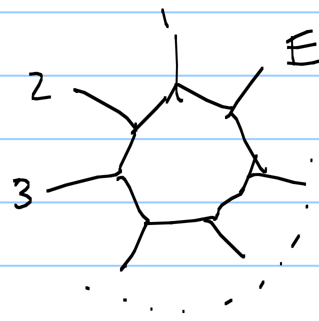
$$\sim \int^{\Lambda} \frac{dl}{l} \sim \log \Lambda$$

diverges when we finally take $\Lambda \to \infty$

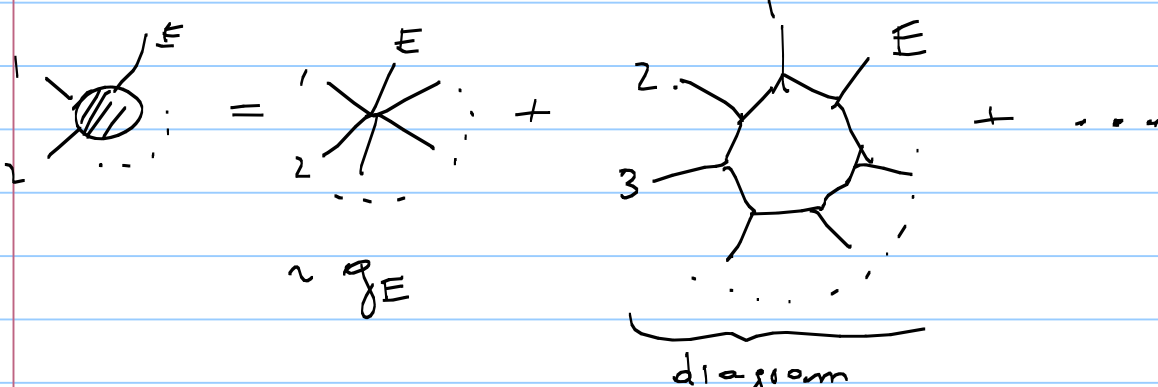
$\therefore$ If $D-1 \geq -1 \rightarrow D \geq 0$ the result tends to be ∞.

There's a more useful way of computing D

Consider



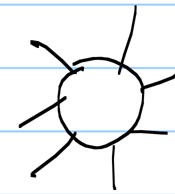
$\rightarrow$ 1-loop diagram



∴ In terms of mass units

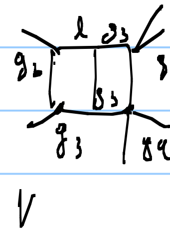
$$[\text{diagram}] = [g_E]$$

But also, for any diagram,



$$[\text{diagram}] = D + \sum_{n=3}^{\infty} V_n [g_n]$$

$\uparrow$
mass dimensions coming from the loop integrations



$$D = [g_E] - \sum_{n=3}^{\infty} V_n [g_n]$$

Remember that we want $D < 0$ for convergence of the integral. Thus, if

$$D \equiv \underbrace{[g_E] - \sum_{n=3}^{\infty} V_n [g_n]}_{\text{positive or zero number}} \geq 0$$

the integral diverges

We see right away that if any $[g_n] < 0$ we get uncontrollable infinities, because D increases with every added vertex of this kind (at higher loops)

∴ $\boxed{\text{A theory with any } [g_n] < 0 \text{ is nonrenormalizable (infinities cannot be canceled out by counter terms)}}$

This criterion, by the way is not always true. A diagram might diverge even if $D<0$ (This could happen when some of the l's in the numerators get cancelled) $\longrightarrow$ QED is an example of this.

Ex:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}m^2\varphi^2 + g_3\varphi^3 + g_4\varphi^4 + g_5\varphi^5 + \cdots$$

$[\mathcal{L}] = M^4 \quad [\partial_\mu] = M \quad \rightarrow \quad [\varphi]^2 M^2 = M^4$

$\therefore [\varphi] = M$

$[g_3]M^3 = M^4 \quad \rightarrow \quad [g_3] = M \quad \longrightarrow \quad [g_5] = -1$

$[g_4]M^4 = M^4 \quad \rightarrow \quad [g_4] = 0 \qquad\qquad [g_6] = -2$

$\therefore$ Only up to φ^4 interaction is renormalizable, i.e. scalar theory

$$\boxed{\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}m^2\varphi^2 + g_3\varphi^3 + g_4\varphi^4}$$

Note: Z_2 invariance gets rid of the φ^3 term.

• Two famous non-renormalizable theories.

i) Fermi's theory of weak interactions:

Its lagrangian looks something like this

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + G(\bar{\psi}\psi)^2$$

(in a simplified form).

Note:

Since $\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} \xrightarrow[\text{Parity}]{} \begin{pmatrix}\psi_R \\ \psi_L\end{pmatrix}$

you can check that $\bar{\psi}\psi \longrightarrow -\bar{\psi}\psi$

thus $(\bar{\psi}\psi)^2$ is parity invariant

but experimentally weak interactions violate parity

Dimensional analysis:

$$M^4 = [\bar{\psi}\gamma^\mu \partial_\mu \psi] = [\bar{\psi}\partial_\mu \psi] = [\psi]^2 M$$

$$\therefore \quad [\psi] = M^{3/2}$$

$$\Longrightarrow [G](M^{3/2 \cdot 2})^2 = M^4 \Rightarrow [G] = M^{-2}$$

$$\therefore \; [G] < 0$$

$\hookrightarrow$ Non-renormalizable

Still, if we try to compute some scattering amplitude, it should look like this

$$M \sim 1 + GE^2 + (GE^2)^2 + \cdots$$

since G is the only parameter of the theory.

Since this theory is non-renormalizable, it still gives good predictions at tree level (low energies). In fact, experimentally

$$G \sim 10^{-5} \, (\text{GeV})^{-2}$$

Thus, theory breaks down at energies E such that

$$GE^2 \sim 1$$ (series doesn't make any sense here)

$$\therefore \; E \sim \frac{1}{\sqrt{G}} \sim 10^2 - 10^3 \text{ GeV}$$

Thus, a new theory of weak interactions is needed and it might contain particles with masses of the order of hundreds of GeV.

Indeed, $SU(2) \times U(1)$ spontaneously broken electroweak predicted and confirmed the existence of $W^{\pm}$, Z gauge bosons with masses 80.4 and 91.2 GeV respectively. $\longrightarrow$ In fact, Fermi's theory can be derived as a low energy effective theory from this one.

ii) Einstein's gravity

For example, in a graviton - graviton scattering, G_N is the only parameter of the theory.

$$V = \frac{G_N M_1 M_2}{r} = [\text{Energy}]$$

$$\frac{[G_N] M^2}{M^{-1}} = M \qquad [r] = M^{-1}$$

$$\therefore \; [G_N] = M^{-2} \longrightarrow \text{Non-renormalizable}$$

$$M \simeq 1 + G_N E^2 + (G_N E^2)^2 + \dots$$

Breaks down at : $E \sim \frac{1}{\sqrt{G_N}} = \underbrace{M_{planck} \cong 10^{19}\ GeV}$

we should expect new physics (or new math?) at these scales.

But certainly, GR gives extraordinary predictions as a low energy effective theory.

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The end

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