This subject is very vast. To try to compress it in only one talk is nearly impossible. I won't go into much details, but rather I will try to explain in more physical terms with simple examples.

In many theories of nature, i.e. theories that have been experimentally verified, the calculation of physical quantities initially gives \( \infty \).

This is the statement of today's discussion.

Examples: QED, Yang-Mills (QCD, Electroweak).

Still, people managed to "understand" those infinities and make extraordinarily accurate predictions (QED \( \rightarrow \) magnetic moment of the electron initially, and many of collider experiments since decades ago).

How does it work?
Let's begin with a toy theory

\[ L = -\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4!} \varphi^4 \]

\[ \varphi \times \varphi : \text{building blocks for interactions} \]

\[ \varphi = i \lambda \]

Let's compute the amplitude of scattering of 4 of these scalar particles:

\[ k_1 \]
\[ k_2 \]
\[ k_3 \]
\[ k_4 \]

\[ = k_1 \times k_4 + k_2 \times k_3 + i \lambda \]

(5)
(6)
(7)

+ ... ...

This is a graphical way of writing

\[ M = i \lambda + \sigma (\lambda^2) \]

tree: all the rest
Let's compute the first term of the rest

\[ \frac{1}{2} (\tilde{\alpha} \lambda)^2 \int \frac{d^4 l}{(2\pi)^4} \frac{-i}{e^2 - m^2 + i\varepsilon} \frac{1}{(l - k_1 - k_3)^2} \]

This gives \(\infty\)

We can actually see that it diverges due to high values of \(e\). For large \(e\)

\[ n \int d\ell \frac{L^3}{\ell^4} = \frac{\Lambda^5}{\ell} = \log \Lambda \ (\Lambda \to \infty) \]

We have a problem right away: the second term in this series expansion should be smaller compared to the first, but it gives \(\infty\)!!

How to make sense of this?

The modern point of view is: We should think of this theory a low energy effective theory of a more complete one.

From this point of view, it makes perfect sense that the cutoff \(\Lambda\) is really present in the theory.
OK!, but now the question is: What value of $\Lambda$ should I use? It seems that the scattering amplitude (i.e., cross sections, and decay rates, etc) directly depend on it.

To answer this, let's compute the integrals:

$$M = -i \lambda + iC \lambda^2 \left[ \log \left( \frac{\Lambda^2}{(K_1 + K_2)^2} \right) + \log \left( \frac{\Lambda^2}{(K_1 + K_3)^2} \right) + \log \left( \frac{\Lambda^2}{(K_2 + K_4)^2} \right) \right]$$

$+ \mathcal{O}(\lambda^3)$

or more concisely

$$\sigma = (K_1 + K_2)^2$$

$\tau = (K_2 + K_3)^2$

$\nu = (K_2 + K_4)^2$

$$M = -i \lambda + iC \lambda^2 \left[ \log \left( \frac{\Lambda^2}{\sigma} \right) + \log \left( \frac{\Lambda^2}{\tau} \right) + \log \left( \frac{\Lambda^2}{\nu} \right) \right]$$

$+ \mathcal{O}(\lambda^3)$

$\lambda$: coupling constant

$\Lambda$: cutoff

Key point: $\lambda$ is NOT the real physical coupling measured in experiments (QED experiments measure $\alpha = e^2 \frac{e}{4\pi}$)
But rather, $\lambda$ and $\lambda'$ are functions of each other.

Experimentally, what one measures is an effective or physical coupling constant $\lambda$.

\[ \begin{align*}
\kappa_1 & \times \kappa_4 = -i \lambda_{\text{phys}} = M \\
\kappa_2 & \times \kappa_3 \\
\text{Log full amplitude or diagrammatic expansion (nature doesn't care about)}
\end{align*} \]

Imagine we go on and measure $\lambda_{\text{phys}}$ by doing experiments at a certain CM energy, i.e.:

\[ \begin{align*}
(k_1 + k_2)^2 &= S_0 \\
(k_2 + k_3)^2 &= t_0 \\
(k_1 + k_4)^2 &= u_0
\end{align*} \]

Thus, our calculation at this energy should give this value of $M = -i \lambda_{\text{phys}}$.

\[ -i \lambda_{\text{phys}} = -i \lambda + i C \lambda^2 \left\{ \log \left( \frac{S_0}{\delta_0} \right) + \log \left( \frac{t_0}{\epsilon_0} \right) + \log \left( \frac{u_0}{\zeta_0} \right) \right\} + o(\lambda^3) \]

or

\[ -i \lambda_{\text{phys}} = -i \lambda + i C \lambda^2 L_0 + o(\lambda^3) \]

Solving for $\lambda$:

\[ i \lambda = i \lambda_{\text{phys}} + i C \lambda^2 L_0 + o(\lambda^3) \]

\[ \text{Plug here correctly} \]
\[ \chi \lambda = i \lambda_{\text{phys}} + i C \lambda_{\text{phys}}^2 L_0 + O(\lambda^3) \]

Thus

\[ M = -i \lambda + i C \lambda_{\text{phys}}^2 L + O(\lambda^3) \]

becomes

\[ M = -i \lambda_{\text{phys}} - i C \lambda_{\text{phys}}^2 L_0 + O(\lambda^3) \]

\[ + i C \left[ i \lambda_{\text{phys}} + i C \lambda_{\text{phys}}^2 L_0 + O(\lambda^3) \right] L + O(\lambda^3) \]

\[ = -i \lambda_{\text{phys}} - i C \lambda_{\text{phys}}^2 L_0 + i C \left[ \lambda_{\text{phys}}^2 + O(\lambda^3) \right] L \]

\[ + O(\lambda^3) \]

\[ M = -i \lambda_{\text{phys}} + \lambda_{\text{phys}}^2 \left( L - L_0 \right) + O(\lambda_{\text{phys}}^3) \]

\[ \Rightarrow L - L_0 = \log \left( \frac{\gamma^2}{S} \right) + \log \left( \frac{\sqrt{t \gamma}}{e} \right) - \log \left( \frac{\gamma^2}{S_0} \right) - \log \left( \frac{\gamma^2}{t_0} \right) - \log \left( \frac{\gamma^2}{u_0} \right) \]

\[ L - L_0 = \log \left( \frac{S_0}{S} \right) + \log \left( \frac{t_0}{t} \right) + \log \left( \frac{u_0}{u} \right) \]
Finally
\[ \eta = -i \chi_{\mu\nu} + \frac{i}{c} \chi_{\mu\nu} \left( \log \left( \frac{s_0}{s} \right) + \log \left( \frac{t_0}{t} \right) + \log \left( \frac{u_0}{u} \right) \right) \]
\[ + \sigma' \chi_{\mu\nu} \]

This is at the heart of Renormalisation.

The dependence on the cut-off \( \Lambda \) has completely disappeared when we express the amplitude in terms of physically observable quantities.

This is what books sometimes call the Miracle of renormalisation.

This 'miracle' does not always happen; it happens for theories that are called Renormalisable.
The Beta function

Our calculation was

\[ M = -i \lambda + i 2^\xi C \left[ \log \frac{N^2}{\xi} + \log \frac{N^2}{\xi} + \log \frac{N^2}{\xi} \right] + o(\lambda^3) \]

The idea of renormalization is that \( \lambda \) and \( \Lambda \) balance each other out in order to yield \( M \) intact under a change of them

\[ \delta M = -i \delta \lambda + 2i \lambda \delta \lambda C \left[ \log \frac{N^2}{\delta} + \ldots \right] \]

\[ + i \lambda^2 C \left[ \frac{2 \times 3 \delta \Lambda}{\Lambda} \right] + o(\lambda^3) \]

\[ \delta \lambda = \delta \lambda \left( -1 + 2 \Lambda C L \right) + 6 \lambda^2 C \frac{\delta \Lambda}{\Lambda} \]

\[ \delta \Lambda = 6 \lambda^2 C + o(\lambda^3) \]
\[ \beta = \frac{d \lambda(n)}{d \ln \lambda} = \frac{d}{d \ln \lambda} \lambda \frac{d \lambda}{d \lambda} \]  

\[ \beta(n) = 6 \lambda^2 c + \delta(\lambda^3) \]

\[ \log \lambda \]

The coupling increases with the energy at which we put the cutoff.

There's of course a more systematic and formal way of computing \( \beta(\lambda) \).

Remark: This 4-point function in the path integral formulation corresponds to the calculation of

\[ \int D \Phi \Phi(x_1) \Phi(x_2) \Phi(x_3) \Phi(x_4) e^{i \int dx \left[ -\frac{1}{2} \partial \Phi \partial \Phi + \frac{1}{4} m^2 \Phi^2 - \frac{A}{4!} \Phi^4 \right]} \]

and restrict \( \int D \Phi \) only to those values of \( \Phi \) whose Fourier transform vanishes for \( k > \lambda \).
Which theories are renormalizable and which are not?

- Infinite expressions come from loop integrals (that's why non-renormalizable theories still give good predictions at the tree level, even though they blow up at the one loop level.)

Consider the diagram:

\[
\begin{array}{c}
\text{in a } \varphi^4 \text{ theory:} \\
\end{array}
\]

\[
\begin{align*}
L &= 1 & V_\gamma &= 4 & \# & \text{of external legs} \\
I &= 4 & V_\delta &= 0 & \# & \text{of internal lines} \\
E &= 4 & V_\nu &= 0 & \# & \text{of closed loops} \\
N_v &= \text{# of vertices that connect } n \text{ lines.}
\end{align*}
\]

\[
\int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + m^2} \frac{1}{(k-k_\gamma)^2 + m^2} \frac{1}{(k-k_\delta)^2 + m^2} \frac{1}{(k-k_\nu)^2 + m^2}
\]

\[
L = 1 \quad V_\gamma = 4 \quad \# & \text{of external legs} \\
I = 4 \quad V_\delta = 0 \quad \# & \text{of internal lines} \\
E = 4 \quad V_\nu = 0 \quad \# & \text{of closed loops}
\]
\[ d \Omega - 2 \Omega \equiv D \quad \text{(superficial degree of divergence)} \]
\[ \int d^D l \cdot l^{D-1} \quad \text{if } D-1 = -1 \]
\[ \sim \int \frac{dl}{l} \quad \text{log } \Lambda \]

diverges when we finally take \( \Lambda \to \infty \)

\[ \text{If } D-1 \geq -1 \Rightarrow D > 0 \text{ the result tends to be } \infty \]

There's a more useful way of computing \( D \)

Consider

\[ \text{1-loop diagram} \]

\[ E = E_1 + E_2 + \ldots \]
\[ \sim \delta E \]

\[ \text{diagram} \]
In terms of mass units

\[ [\text{diagram}] = [\phi] \]

But also, for any diagram,

\[
[\text{diagram}] = D + \sum_{m=3}^{\infty} V_m [g_m]
\]

mass dimensions coming from the loop integrations

\[ D = [\phi] - \sum_{m=3}^{\infty} V_m [g_m] \]

Remember that we want \( D \leq 0 \) for convergence of the integral. Thus if

\[ D = [\phi] - \sum_{m=3}^{\infty} V_m [g_m] > 0 \]

the integral diverges

We see right away that if any \( [g_m] < 0 \) we get uncontrolled infinities. Because \( D \) increases with every added vertex of this kind (at higher loops)

A theory with any \( [g_m] < 0 \) is nonrenormalizable (infinities cannot be canceled out by counter terms)
This criterion, by the way, is not always true. A diagram might diverge even if \( D \leq 0 \) (this could happen when some of the \( \lambda \)'s in the numerators get cancelled) \( \rightarrow \text{QED is an example of this.} \)

\[
L = -\frac{1}{2} \partial \psi \partial \psi^{\dagger} + \frac{i}{2} m^2 \psi^{\dagger} \psi + g_2 \psi^{\dagger} \psi + g_4 \psi^{\dagger} \psi^{\dagger} \psi^2 + \ldots
\]

\[\lambda^2 = M^4 \quad \lambda^4 = M \quad \rightarrow \quad [\lambda^4]_2 \quad M^2 = M^4 + \ldots
\]

\[\therefore \quad [\psi^4] = M
\]

\[\left[ g_2 \right] M^3 = M^\nu \quad \rightarrow \quad \left[ g_2 \right] = M \quad \rightarrow \quad [g_2] = -1
\]

\[\left[ g_4 \right] M^5 = M^\nu \quad \rightarrow \quad [g_4] = 0 \quad \left[ g_6 \right] = -2
\]

\[\therefore \quad \text{Only up to} \quad \psi^4 \quad \text{interaction is renormalizable, i.e. scalar theory}
\]

\[
L = -\frac{1}{2} \partial \psi \partial \psi^{\dagger} + \frac{i}{2} m^2 \psi^{\dagger} \psi + g_2 \psi^{\dagger} \psi + g_4 \psi^{\dagger} \psi^{\dagger} \psi^2
\]

\[\text{Note: \( Z_2 \) invariance gets rid of the} \quad \psi^3 \quad \text{term.}\]
Two famous non-renormalizable theories.

1) Fermi's theory of weak interactions:
   The Lagrangian looks something like this:
   \[ L = \mathcal{L}_1 \left( i \gamma^\mu \partial_\mu - m \right) \psi + G \left( \overline{\psi} \gamma^\mu \psi \right)^2 \]
   (in a simplified form).

   Note:
   Since \( (\psi_L) \rightarrow (\psi_R) \) under parity,
   you can check that \( \overline{\psi} \gamma^\mu \psi \rightarrow -\overline{\psi} \gamma^\mu \psi \)
   Thus \( (\overline{\psi} \gamma^\mu N \psi)^2 \) is parity invariant,
   but experimentally weak interactions violate parity.

   Dimensional analysis:
   \[ M^y = \left[ \overline{\psi} \gamma^\mu N \psi \right] = \left[ \overline{\psi} \gamma^\mu N \psi \right] = [4J]^2 \cdot M \]
   \[ \therefore [4J] = M^{3/2} \]
   \[ \Rightarrow [GJ (M^3/N^2)^2] = M^y \Rightarrow [GJ] = M^{-2} \]
\[ \therefore [G] < 0 \]

\[ \rightarrow \text{Non-renormalizable} \]

Still, if we try to compute some scattering amplitude, it should look like this

\[ M \sim 1 + GE^2 + (GE^2)^2 + \ldots \]

since \( G \) is the only parameter of the theory.

Since this theory is non-renormalizable, it still gives good predictions at tree level (low energies). In fact, experimentally

\[ 6 \sim 10^{-6} \text{ (GeV)}^2 \]

Thus, theory breaks down at energies \( E \) such that

\[ GE^2 \sim 1 \] (sense doesn't make any sense here)

\[ \therefore E \sim \frac{1}{\sqrt{6}} \sim 10^2 - 10^3 \text{ GeV} \]
Thus, a new theory of weak interactions is needed and it might contain particles with masses of the order of hundreds of GeV.

Indeed, SU(2) \times U(1) spontaneously broken electroweak predicted and confirmed the existence of W\text{±}, Z gauge bosons with masses 80.4 and 91.2 GeV respectively. In fact, Fermi's theory can be derived as a low energy effective theory from this one.

**Einstein's Gravity**

For example, in a graviton-graviton scattering, \( G_N \) is the only parameter of the theory.

\[
V = \frac{G_N M_1 M_2}{r} = \text{(Energy)}
\]

\[
\frac{[G_N]}{M^2} = M \quad \text{(r)} = M^{-1}
\]

\[
\therefore [G_N] = M^{-2} \quad \text{non-renormalizable}
\]
\[ M \approx 1 + 6N E^2 + (6N E^2)^2 + \ldots \]

Breaks down:

\[ E \sim \frac{1}{\sqrt{6N}} \Rightarrow M_{\text{planck}} \approx 10^{19} \text{ GeV} \]

we should expect new physics (or new math?) at these scales.

But certainly GR gives extraordinary predictions as a low energy effective theory.

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The end

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