Gravitational Waves

sig: $-1, 1, 1, 1$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Einstein's Equation(s)

$R_{\mu\nu}$ - Ricci tensor, (curvature) $\quad R$ - $R^{\mu}{}_{\mu}$ Ricci's scalar

$T_{\mu\nu}$ - source stress energy density

We begin by perturbing flat space (ripples)

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h| \ll 1$ (linearized theory from nonlinear GR)

(gloss over the loss of gauge symmetry)

Compute the Riemann tensor: $R_{\mu\nu\rho\sigma}$

in its full glory:

$$R_{\mu\nu\rho\sigma} = \eta^{\mu}{}_{\lambda} R^{\lambda}{}_{\mu\nu\rho\sigma} = \eta^{\mu}{}_{\lambda} \partial_{\rho} \Gamma^{\lambda}{}_{\sigma\nu} + \eta^{\mu}{}_{\lambda} \partial_{\nu} \Gamma^{\lambda}{}_{\rho\sigma}$$

$+$

$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu} \Gamma^{\rho}{}_{\nu\sigma}$

$\cancel{R^{\rho}{}_{\sigma\mu\nu}}$

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\sigma\nu} - \partial_\sigma \Gamma^{\mu}_{\rho\nu} + \Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}_{\sigma\nu}$$

$$- \Gamma^{\mu}_{\sigma\lambda}\Gamma^{\lambda}_{\rho\nu}$$

Encodes all information about the curvature of a space

~~Note~~

Γ ? $\Big]$ Γ is the connection, it makes the derivative into an actual tensor (correct transformation properties)

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$$

In field theories $g^{\lambda\sigma} = \eta^{\lambda\sigma}$ and all derivatives vanish so the c. derivative

$$\nabla_\mu V^\mu = \partial_\mu V^\nu + \Gamma^{\nu}_{\mu\sigma} V^\sigma \equiv \partial_\mu V^\nu \quad \text{(lucky guys)}$$

In our case, this is no longer true.

However, first note that $\Gamma \sim \mathcal{O}(h)$

So $\Gamma\Gamma = \mathcal{O}(h^2)$ and can be dropped

Leaving us with only

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu{}_{\sigma\nu} - \partial_\sigma \Gamma^\mu{}_{\rho\nu}$$

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let's look at $\Gamma^\mu{}_{\sigma\nu}$ for our metric:

$$\Gamma^\mu{}_{\sigma\nu} = \frac{1}{2} g^{\mu\lambda} \left(\partial_\sigma g_{\nu\lambda} + \partial_\nu g_{\lambda\sigma} - \partial_\mu g_{\sigma\nu} \right)$$

$$g = \eta^{\mu\lambda} + h^{\mu\lambda} \qquad \partial g = \partial h$$

$$\Gamma^\mu{}_{\sigma\nu} = \frac{1}{2} \cancel{g} \left(\eta^{\mu\lambda} + h^{\mu\lambda} \right) \left(\partial_\sigma h_{\nu\lambda} + \partial_\nu h_{\lambda\sigma} - \partial_\mu h_{\sigma\nu} \right)$$

but $h\,(O(h)) = O(h^2)$ (too small)

$$= \frac{1}{2} \eta^{\mu\lambda} \left(\partial_\sigma h_{\nu\lambda} + \partial_\nu h_{\lambda\sigma} - \partial_\mu h_{\sigma\nu} \right)$$

So the Riemann tensor is essentially all 2nd derivatives of the perturbation. After a lot of math!

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} \left(\partial_\nu \partial_\rho h_{\mu\sigma} + \partial_\mu \partial_\sigma h_{\nu\rho} - \partial_\mu \partial_\rho h_{\nu\sigma} - \partial_\nu \partial_\rho h_{\mu\rho} \right)$$

(two terms cancelled in the math)

But we can make this <u>prettier</u>

define

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

$$\Rightarrow h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \quad \text{(trust me)}$$

this simplifies things a bit:

$$\cancel{R_{\mu\nu\rho\sigma} = \left(\frac{1}{2}\partial_\nu \partial_\rho\right.}$$

$$2R_{\cancel{\mu\nu}} = \partial^\nu \partial^\rho \bar{h}_{\rho\nu} + \partial^\nu \partial^\rho \bar{h}_{\nu\rho} + \partial^2 \bar{h}$$

$$2R_{\mu\nu} = \partial_\mu \partial^\rho \bar{h}_{\rho\nu} + \partial_\nu \partial^\rho \bar{h}_{\mu\rho} - \partial^2 \bar{h}_{\mu\nu} + \frac{\eta_{\mu\nu}}{2}\partial^2 \bar{h}$$

and after all:

$$\partial_\mu \partial^\rho \bar{h}_{\rho\nu} + \partial_\nu \partial^\rho \bar{h}_{\mu\rho} - \partial^2 \bar{h}_{\mu\nu} - \eta_{\mu\nu}\partial^\sigma \partial^\rho \bar{h}_{\rho\sigma} = \frac{16\pi G}{c^4} T_{\mu\nu}$$

But what do we do with this?

It has some symmetries, but isn't very useful. $\partial_\mu \partial^\rho$ are ugly.

We still have some freedom inherent in a gauge choice. Since this is a matter of really homogeneous 2nd derivatives, it is a possibility to massage this into a wave equation. This motivates the ~~Lorentz~~ Loren(t)z gauge:

$$\partial^\nu \bar{h}_{\mu\nu} = 0 \qquad \text{c.f.} \quad \boxed{\partial^\mu A_\mu = 0}$$

POOF! 3 terms disappear instantaneously!

and we are left w/

$$\partial^2 \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

a true wave equation w/ sources

Now what? Let's look at the free space solution (i.e. $T_{\mu\nu} = 0$)

$$\partial^2 \bar{h}_{\mu\nu} = 0$$

free space tensor wave equation

Let's look closer at this. We can be more specific. Let's say the time index

$$\partial^0 h_{00} + \partial^i h_{0i} = 0$$

by defin of above

~~Then~~

Translational invariance:

$x'^{\mu} = x^{\mu} + \xi^{\mu}$ implies that we can further impose (w/o proof)

$$h^{0i} = 0 \quad \& \quad \partial^0 h_{00} + \partial^i h_{0i} = 0$$

$$\Rightarrow h_{00} = 0$$

altogether: $h^{0\mu} = 0$ (4 DOF)

We can also impose that the trace vanishes.

$$h^{\mu}_{\ \mu} = 0$$

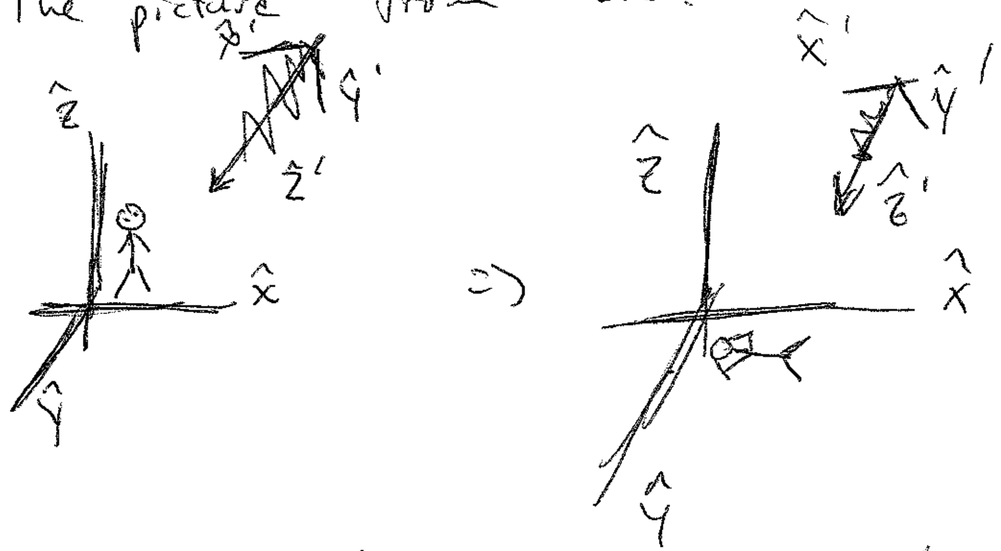
Together this is the <u>transverse</u>-<u>traceless</u> gauge and gives us a wave w/ two independent components that behave very much like a E/M wave (though E & B are not independent)

Formally, the solution:

$$h_{ij}^{TT}(x) = \hat{e}_{ij}(\vec{k})\, e^{ikx}$$

where the $\hat{e}_{ij}$ is a polarization tensor

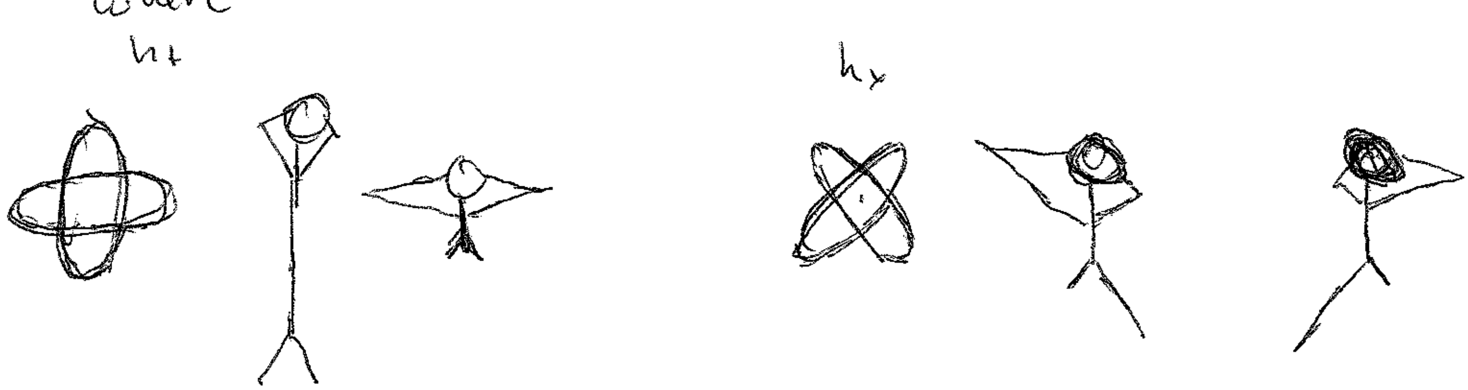
The picture from here:



$$h_{ij}^{TT}(t,z) = \underbrace{\begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{two independent polarizations}} e^{\underline{i(\omega(t - z/c))}}$$

usual wave stuff

where:



Spin-2 graviton

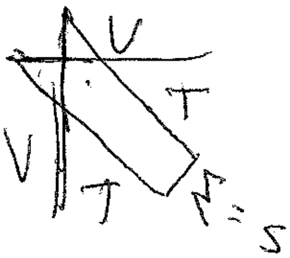
Follows Maggiore (2008) chp 2 pg 66-81

"Proof" by negation. Try all the possibilities

Spin $\emptyset$:

$\Rightarrow$ scalar field φ

Decompose the tensor field into its S/V/T parts



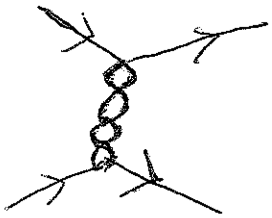
Spin 0 $\rightarrow$ no indices ... must be the trace of E-M tensor ... $T^{\mu}{}_{\mu} = T$

write our Lagrangian

$$\mathcal{L} = \underbrace{-\tfrac{1}{2}\left(\partial_\mu \varphi \partial^\mu \varphi + m^2 \varphi^2\right)}_{\text{scalar Lagrangian}} + g\varphi T \quad \boxed{\text{coupling term}}$$

The check? Does it reduce to Newtonian gravity in the nonrelativistic limit?

So we calculate the "induced potential" at the exchange of a graviton, consider tree level:



$\Rightarrow$

$D(q) = \frac{-i}{q^2 + \mu^2}$

S. field propagator

we use

$$V(\vec{x}) = -\int \frac{d^3 q}{(2\pi)^3} \, M(\vec{q}) \, e^{i\vec{q}\cdot\vec{x}}$$

$M(\vec{q})$: scattering amplitude

non-relativistically speaking:

$$iM(\vec{q}) = (-ig)^2 \, \tilde{T}_1(\vec{q}) \, \tilde{D}(\vec{q}) \, \tilde{T}_2(-\vec{q})$$

$\tilde{T}$ external field E-M tensor

in the static limit $q^2 = q_0^2 + \vec{q}^{\,2} \cong \vec{q}^{\,2}$ (static limit)

$$T^{\mu\nu}(\vec{x}, t) = \frac{p^\mu p^\nu}{p^0} \, \delta(\vec{x} - \vec{x}_0(t))$$ classical particle on trajectory $\vec{x}_0(t)$

in the static limit:

$$p^\mu p_\mu = -m^2 \qquad p^0 = m$$

so $T(\vec{x},t) = -m\,\delta(\vec{x} - \vec{x}_0(t))$

$\Rightarrow \tilde{T}(\vec{q}) = -m$

altogether:

$$V(\vec{x}) = -ig^2 \int \frac{d^3q}{(2\pi)^3} \tilde{T}_1(\vec{q}) \frac{-i}{\vec{q}^2 + \mu^2} \tilde{T}_2(-\vec{q})\, e^{i\vec{q}\cdot\vec{x}}$$

$$\text{or} \quad = -ig^2 \int \frac{d^3q}{(2\pi)^3} \; +(m_1 m_2) \; \frac{-i}{\vec{q}^2 + \mu^2}\, e^{i\vec{q}\cdot\vec{x}}$$

$$= -g^2 m_1 m_2 \int \frac{d^3q}{(2\pi)^3} \frac{e^{i\vec{q}\cdot\vec{x}}}{\vec{q}^2 + \mu^2}$$

infinite range $\Rightarrow \mu^2 = 0$

$$= -g^2 m_1 m_2 \left(+ 1/4\pi r \right)$$

$$= -\frac{g^2}{4\pi} \frac{m_1 m_2}{r} \qquad G = \frac{g^2}{4\pi}$$

$$= -\frac{G m_1 m_2}{r}$$

... it seems to work!

But wait ... let's look at a simple test case: (9)

$$T_{\mu\nu} = F^{\mu\nu} F_{\mu\nu} - \frac{1}{4} \delta^{\mu}{}_{\nu} F^2$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E & & \\ & & 0 & B \\ -E & -B & 0 & 0 \end{pmatrix}$$

... is antisymmetric

... is traceless ... !

$\int \varphi T$ ... $= 0$!

But gravity bends light.

$\boxed{\text{NEXT!}}$

Case: spin 1

coupling terms? :

$\boxed{A_\mu j^\mu \quad - \text{possible}}$

$A_\mu A_\nu T^{\mu\nu}$ - not gauge invariant

$\partial_\mu A_\nu T^{\mu\nu}$ - $\emptyset$ $\quad \partial^\mu T_{\mu\nu} = 0$ in flat space

interpret as mass density

$$\Rightarrow V(\vec{x}) = -i g^2 \int \tilde{j}^\mu(\vec{q}) \tilde{D}_{\mu\nu}(\vec{q}) \tilde{j}^\nu(\vec{q}) \frac{d^3 q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}}$$

for a massless vector field $A^{\mu}(q)$

$$\tilde{D}_{\mu\nu}(q) = \frac{-i}{q^2}\eta_{\mu\nu} \Rightarrow \frac{-i\,\eta_{\mu\nu}}{\vec{q}^{\,2}}$$

$$V(\vec{x}) = -\tfrac{\hbar}{2}g^2 \int \frac{d^3q}{(2\pi)^3}\, \tilde{j}^{\mu}(\vec{q})\, \tilde{j}_{\mu}(-\vec{q})\, \frac{1}{\vec{q}^{\,2}}\, e^{i\vec{q}\cdot\vec{x}}$$

$$\tilde{j}^{\mu}(q) = (\rho, \vec{0})$$

~~$j^{\mu}j_{\mu} = -\rho^2$~~

$j^{\mu}j_{\mu} = j^{0}j_{0}$

$= -m_1 m_2$

$$= g^2 \frac{m_1 m_2}{r}$$

... ~~but gravity isn't repulsive~~

~~telling~~

... but you try and tell gravity its repulsive...

its insulting...

How about spin ≥ 2 ?

Massless fields couple to conserved tensors. (???)

There is practically no tensor conserved w/ > 2 indices.

So ... no.

4-vect $A_\mu \in 0 \oplus 1$ $(\underbrace{t}_{}, \underbrace{x, y, z}_{})$

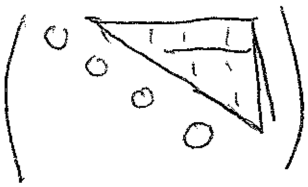
$\Uparrow$ Scalar $\quad \Uparrow$ SO(3)

SO(3,1) Lorentz invariant

antisymmetric tensor

$A_{\mu\nu} \in 1 \oplus 1$

SO(3) × SO(3)



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Symmetric (traceless) tensor

$S_{\mu\nu} \in 0 \oplus 1 \oplus 2$

spin representation : $2s+1 = N_{(\dim)}$

$A^\mu \Rightarrow 2s+1 = 3$ $\quad s = 1$ | Scalar $\quad 2s+1 = 1$

SO(3) $\quad$ u(1)

$s = 0$

$A^{\mu\nu}$ $\quad 2s+1 = 6 \Rightarrow$ ~~$s = 5/2$~~ ?

SO(3) × SO(3)

$2s+1 = 5$ $\quad S = 1$

$S_{\mu\nu}$ contains 2 (we've eliminated everything else) ①

$N_{(comp)} = 2s+1 = 5$ (spin 2) ②

$2s+1 = 3$ (spin 1) ①

$2s+1 = 1$ (spin 0) ⓪

9 independent components

+1 for the addition of the trace

Recall again that we require massless representations where we have only 2 DOF (helicity $\pm s$)

But we have <u>NINE</u>!

Let's take a lesson from history. How do we make a photon from A^μ?

(Jackson 240)

gauge invariance

$$A_\mu \rightarrow A_\mu - \partial_\mu \theta = \left\{ \theta \left\{ \begin{array}{l} \nabla \cdot \vec{A} + \frac{\partial \varphi}{\partial t} = 0 \\ \nabla \cdot \vec{A} = -\frac{\partial \varphi}{\partial t} \end{array} \right. \right.$$

or $\boxed{\partial^\mu A_\mu = 0}$

or $\nabla \cdot \vec{A} = 0$ Coulomb gauge

$+ \frac{\partial \varphi}{\partial t}$ no sources

2 conditions

4 DOF

− 2 conditions (gauge)

2 DOF ± 1 ~~polarization states~~ helicities

So what gauge do we choose for the graviton
... a simple generalization

$$h_{\mu\nu} \Rightarrow \underset{\substack{\Uparrow \\ \text{symmetric}}}{h_{\mu\nu}} - \underset{\substack{\Uparrow \\ \text{also symmetric}}}{(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)}$$

(why?)

Given this, how do we construct an action consistent w/ everything we've said

- symmetric trace free tensor ($h_{\mu\nu}$)
- gauge invariance

w/ additional "guess" quadratic in the fields (think φ^2 or $(\nabla\varphi)^2$) w/ 2 derivatives (why?)

Consider all the terms we can construct

$\partial_\rho h^{\mu\nu} \partial^\rho h_{\mu\nu}$

$$\left.\begin{array}{l} \partial_\rho h_{\mu\nu} \partial^\nu h^{\mu\rho} \\ \partial_\nu h^{\mu\nu} \partial^\rho h_{\mu\rho} \end{array}\right\} \Rightarrow \text{related}$$

(extra credit for how)

$\partial_\nu h^{\mu\nu} \partial_\mu h$

$\partial^\mu h \, \partial_\mu h$

$$S = \frac{1}{2}\int d^4x \left(-\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + 2\partial_\rho h_{\mu\nu}\partial^\nu h^{\mu\rho} - \partial_\nu h^{\mu\nu}\partial_\mu h + \partial^\mu h \partial_\mu h \right)$$

Pauli - Fierz Action

Check w/ E-L equations:

dL

$$\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu h_{\alpha\beta})}} - \underbrace{\frac{\partial \mathcal{L}}{\partial h_{\alpha\beta}}}_{= 0} \quad \text{(true?)}$$

↳

$$-\partial^\sigma \partial_\rho h_{\mu\nu} \quad \Rightarrow \quad \boxed{\nabla^2 h_{\mu\nu}} \; ?$$

this works, so fixing gauges gives us GW (a spin 2 field now)

So pick the gauges which get us

$\partial^\mu h_{\mu\nu} = 0$ (Lorentz gauge - 4 conditions)

$\cancel{\Box^2 h_{\mu\nu} \to T}$

$$\Box^2 h_{\mu\nu} \to \Box^2 (h_{\mu\nu} + \partial_\nu \xi_\mu + \partial_\mu \xi_\nu)$$

$$\Rightarrow \nabla^2 \xi_\mu \quad \text{(4 more conditions)}$$

9 + 1 parameters
- 4 Lorentz
- 4 residual gauge

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2 DOF ✓

Good, its what we expect. But does it reduce to the correct limit?

Propagator for a symmetric ~~spin 2~~ tensor field:

$$\tilde{D}_{\mu\nu\rho\sigma}(k) = \frac{1}{2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\right)\left(\frac{-i}{k^2 - i\epsilon}\right)$$

Static limit ⇒ energy density → potential field

look at D_{0000}

$$\tilde{D}_{0000}(k) = \frac{1}{2}\left(1 + 1 - 1\right)\left(\frac{-i}{k^2 - i\varepsilon}\right)$$

$$= \frac{-i}{2k^2}$$

Same as scalar which we <u>know</u> <u>works</u>!

But all of this is linear

→ GR is <u>non</u>-linear

In order to consider everything including an interaction term:

$$\int h_{\mu\nu} T^{\mu\nu} d^4x$$

we break our E-M conservation theorem

i.e.

$$\partial^\mu T_{\mu\nu} \neq 0 \quad \underline{\text{inside matter}}$$

and therefore we need

$$T^{\mu\nu} \to \underset{\text{matter}}{T^{\mu\nu}} + \underset{\text{GW}}{t^{\mu\nu}}$$

but then we have

$$h^{\mu\nu} \mathbb{T}^{\mu\nu} = h^{\mu\nu} T^{\mathrm{tot}}_{\mu\nu} + h^{\mu\nu} t_{\mu\nu}$$

where $t_{\mu\nu}$ is a function of $h_{\mu\nu}$

... which we have to add back into the action.

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... this ~~???~~ gives us recursively higher & higher order terms and represents the self interaction and the non-linearity of GR.