Gravitational Waves

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \]

Einstein's Equation (8)

\( R_{\mu\nu} - \text{Ricci tensor, } R = R_{\mu\nu} \text{ Ricci scalar (curvature)} \)

\( T_{\mu\nu} = \text{source stress energy density} \)

We begin by perturbing flat space (ripples)

\[ g_{\mu\nu} \approx \eta_{\mu\nu} \text{ then } |h| \ll 1 \text{ (linearized theory from nonlinear GP)} \]

(gloss over the loss of gauge symmetry)

Compute the Ricci tensor: \( R_{\mu\nu} \)

in its full glory:

\[ R_{\mu\nu} = \frac{\partial \Phi_{\mu\nu}}{\partial x^\rho} + \frac{\partial \Phi_{\rho\mu}}{\partial x^\nu} - \frac{\partial \Phi_{\rho\nu}}{\partial x^\mu} + \frac{\partial \Phi_{\nu\rho}}{\partial x^\mu} - \frac{\partial \Phi_{\mu\rho}}{\partial x^\nu} \]
\[ R^\mu_{\nu\sigma\tau} = \partial_\nu \Gamma^\mu_{\sigma\tau} - \partial_\sigma \Gamma^\mu_{\nu\tau} + \Gamma^\mu_{\nu\lambda} \Gamma^\lambda_{\sigma\tau} - \Gamma^\mu_{\sigma\lambda} \Gamma^\lambda_{\nu\tau} \]

Encodes all information about the curvature of a space

\[ \Gamma^\lambda_{\mu\nu} \]

\( \Gamma \) is the connection, it makes the derivative into an actual tensor (correct transformation properties)

\[ \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left( \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right) \]

In field theory, \( g^{\mu\nu} = \eta^{\mu\nu} \) and all derivatives vanish so the C derivative

\[ \nabla_\nu \eta^\mu = \partial_\nu \eta^\mu + \Gamma^\mu_{\nu\sigma} \eta^\sigma = \partial_\nu \eta^\mu \text{ (lucky guys!)} \]

In our case, this is no longer true. However, first note that \( \Gamma \sim O(h) \)

So \( \nabla_\nu \eta^\mu \sim O(h^2) \) and can be dropped
Leaving us with only

\[ R^a_{\mu \nu \rho \sigma} = \partial_\rho R^a_{\mu \nu \sigma} - \partial_\sigma R^a_{\mu \nu \rho} \]

let's look at \( R^a_{\mu \nu} \) for our metric:

\[ R^\mu_{\nu \rho \sigma} = \frac{1}{2} \Gamma^\mu_{\nu \rho} \Gamma^\rho_{\nu \sigma} + \partial_\nu g_{\rho \sigma} - \partial_\rho g_{\nu \sigma} \]

\[ g = \eta + h \quad \delta g = \delta h \]

\[ R^\mu_{\nu \rho \sigma} = \frac{1}{2} g \left( \eta^{\mu \rho} + h^{\mu \rho} \right) \left( \partial_\nu h_{\sigma} + \partial_\sigma h_{\nu} - \partial_\nu \delta h_{\sigma} - \partial_\sigma \delta h_{\nu} \right) \]

but \( h (0(h) = 8h^2) \) (too small)

\[ = \frac{1}{2} \eta^{\mu \rho} \left( \partial_\nu h_{\sigma} + \partial_\sigma h_{\nu} - \partial_\nu \delta h_{\sigma} - \partial_\sigma \delta h_{\nu} \right) \]

So the Riemann tensor is essentially all 2nd derivatives of the perturbation. After a lot of work:

\[ R^a_{\mu \nu \rho \sigma} = \frac{1}{2} \left( \partial_\rho \partial_\nu h_{\mu \sigma} + \partial_\sigma \partial_\nu h_{\rho \mu} - \partial_\rho \partial_\mu h_{\sigma \nu} \right) - \partial_\nu \partial_\rho h_{\mu \sigma} \]

(two terms cancelled in the work)
But we can make this prettier

define

\[ \tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \tilde{h} \]

\[ \tilde{h}_{\mu\nu} = \tilde{h}_{\nu\mu} - \frac{1}{2} \eta_{\mu\nu} \tilde{h} \quad \text{(trust me)} \]

this simplifies things a bit:

\[ R_{\mu\nu} = \frac{1}{2} T_{\mu\nu} \]

2 \[ R_{\mu\nu} = \partial^\nu \partial^\mu \tilde{h} - \partial^\mu \partial^\nu \tilde{h} + \partial^2 \tilde{h} \]

2 \[ R_{\mu\nu} = \partial_\mu \partial_\nu \tilde{h} - \partial_\nu \partial_\mu \tilde{h} + \partial^2 \tilde{h} + \frac{\eta_{\mu\nu}}{2} \partial^2 \tilde{h} \]

and after all:

\[ \partial_\mu \partial^\mu \tilde{h} - \partial_\nu \partial^\nu \tilde{h} = \partial^2 \tilde{h} + \frac{\eta_{\mu\nu}}{2} \partial^2 \tilde{h} \]

\[ = \frac{16 \pi G}{c^4} T_{\mu\nu} \]

But what do we do with this?

It has some symmetries but isn’t very useful. \( \partial_\mu \partial^\mu \tilde{h} \) are ugly.
We still have some freedom inherent in a gauge choice. Since this is a sum of very homogeneous and derivable, it is a possibility to massage this into a wave equation. This requires the Lorentz \( \sigma \) gauge:

\[ \nabla^2 \mathcal{T}_{\mu\nu} = 0 \quad \text{c.f.} \quad \Delta^2 \mathcal{A}_\mu = 0 \]

\( \sim \) Proof! Sources disappear instantaneously!

and we are left with:

\[ \Delta^2 \mathcal{T}_{\mu\nu} = - \frac{16\pi G}{c^4} \mathcal{T}_{\mu\nu} \quad \text{a true wave equation of source} \]

Now what? Let's look at the free space solution (i.e. \( \mathcal{T}_{\mu\nu} = 0 \)).

\[ \Delta^2 \mathcal{T}_{\mu\nu} = 0 \quad \text{free space tensor wave equation} \]

Let's look closer at this! We can be more specific: Let's say \( \text{th time index} \)

\[ \partial_0 \mathcal{H}_{00} + \mathcal{D}_i \mathcal{H}_{0i} = 0 \quad \text{by definition of above} \]
Translational invariance:

\( x' = x + \delta x \) implies that we can further impose (w/o proof)

\[ h^{0i} = 0 \quad \Rightarrow \quad h^{00} = 0 \]

altogether: \( h^{00} = 0 \) \((4 \text{ DoF)}\)

We can also impose that the trace vanishes:

\( \bar{h}^{\mu \nu} = 0 \)

Together this is the \underline{transverse - traceless gauge} and gives us a wave of two independent components that behave very much like a \( \text{ElM} \) wave (though \( E \& B \text{ are not independent} \))
Formally, the solution:

\[ h_{ij}(x) = \hat{e}_{ij}(F) e^{i k x} \]

where the \( \hat{e}_{ij} \) is a polarization

The picture from here:

\[ \hat{e} \]

\[ \begin{align*}
    h_{ij}(t, x) &= \begin{pmatrix}
        h_t & h_x & 0 \\
        h_x & -h_t & 0 \\
        0 & 0 & 0
    \end{pmatrix} e^{i \omega (t - z/c)} \\
    \text{two independent} \\
    \text{polarizations}
\end{align*} \]

where:

- \( h_t \)
- \( h_x \)
Follows Maggiore (2003) chap 2 pg 66-81

"Proof" by negation. Try all the possibilities

Spin $\phi$

$\Rightarrow$ Scalar field $\phi$

Decompose the tensor field into its $S/V/T$ parts

Spin 0 is no indices... must be the trace of

E-M tensor... $T^{\mu \nu} = T$

write our Lagrangian

$$L = \frac{1}{2} (\partial \phi \cdot \partial \phi + m^2 \phi^2) + g \phi^4$$

Scaling Lagrangian

The check? Does it reduce to Newtonian gravity

in the nonrelativistic limit?
So we calculate the "induced potential" at the exchange of a graviton. Consider the level:

\[ D(q) = \frac{-i}{q^2 + m^2} \]

5. Field propagator

we use

\[ V(\vec{x}) = -\int \frac{d^3q}{(2\pi)^3} M(q) e^{i\vec{q} \cdot \vec{x}} \]

scattering amplitude

non-relativistically speaking:

\[ M(q) = (-i\hbar)^2 \frac{\tau_3}{15} \bar{\sigma}(\bar{q}) D(q) \bar{T}_z (-\bar{q}) \]

external field E-M tensor

in the static limit \( q' = q_0 + \bar{q}' \Rightarrow \bar{q}'^2 \) (static)

\( T^\mu_\nu(\vec{x}, t) = \frac{p_\mu p_\nu}{p^0} \delta(\vec{x} - \vec{x}_0(t)) \) (classical particle trajectory \( R_{0t} \))
In the static limit:

\[ p^μ p_μ = -m^2 \quad \Rightarrow \quad p_μ = m \]

So

\[ \Gamma (\vec{x}, t) = -m \delta (\vec{x} - \vec{x}_0 (t)) \]

\[ \Rightarrow \Gamma (\vec{q}) = -m \]

altogether:

\[ V (\vec{x}) = -2g^2 \int \frac{d^3q}{(2\pi)^3} \left[ \frac{-i}{\vec{q}^2 + m^2} + \frac{1}{2} \right] e^{i\vec{q} \cdot \vec{x}} \]

or

\[ = -\frac{2g^2}{(2\pi)^3} \left[ \frac{-i}{\vec{q}^2 + m^2} + (m_1 M_2) \frac{-i}{\vec{q}^2 + m_2^2} \right] e^{i\vec{q} \cdot \vec{x}} \]

\[ = \frac{-g^2 M_1 M_2}{(2\pi)^3} e^{i\vec{q} \cdot \vec{x}} \]

\[ = \frac{-g^2 M_1 M_2}{(2\pi)^3} \right( \frac{\lambda^2}{4\pi} \right) \]

\[ = -\frac{g^2 M_1 M_2}{(2\pi)^3} \frac{\lambda^2}{4\pi} \]

\[ C = \frac{g^2}{4\pi} \]

\[ \Rightarrow \frac{GM_1 M_2}{r} \]

... it seems to work!
But wait... let's look at a simple test case: \( T_{\mu\nu} = F_{\mu\lambda} F_{\nu}^{\lambda} - \frac{1}{4} g_{\mu\nu} F^2 \)

\[
F_{\mu\nu} = \begin{pmatrix}
0 & E \\
-E & 0
\end{pmatrix}
\]

is antisymmetric

is traceless

\( \nabla \phi + \cdots = 0 \)

But gravity bends light.

**NEXT**

case: spin 1

simplify terms?

\[ \text{An} \] is possible

\[ \text{An} \] is not gauge invariant

\[ 2 \text{An} \] is \( \emptyset \)

in flat space

\[ V(\phi) = -i g^2 \sum j_{\mu}(q) \hat{D}^\mu(\overline{\psi}) \overline{\psi}(\psi) \frac{d^3q}{(2\pi)^3} \]
for a massless vector field $A^{\mu}(q)$

$$\hat{D}_{\nu} \phi = \frac{i}{q^2} \nabla_{\nu} = \frac{i}{q^2} \frac{2}{q^2}$$

$$V(r) = -\frac{a}{2} \sum_{\lambda} \int \frac{d^3q}{(2\pi)^3} \, \hat{j}^{\lambda}(q) \hat{j}^{\bar{\lambda}}(q)$$

$$\hat{j}^\lambda(q) = \left( \begin{array}{c} \xi \\ \phi \end{array} \right)$$

$$\hat{j}^\lambda \hat{j}_{\lambda} = \int \frac{d^3q}{(2\pi)^3} \left( \begin{array}{c} \xi \\ \phi \end{array} \right) \left( \begin{array}{c} \xi \\ \phi \end{array} \right) = -\mu \cdot \mu$$

...but gravity isn't repetitive

...but you can't tell gravity it's repetitive...

...it's insulting...

How about $Spin \geq 2$?

Massless fields couple to conserved tensors.

There is practically no tensor conserved except $2$ indices.

So... no.
\[ A_{\mu} \in 0 \oplus 1 \]

\[ \uparrow \quad \uparrow \]

Scalar \quad SO(3)

\[ SO(3,1) \quad Lorentz \; invariant \]

\text{antisymmetric tensor}

\[ A_{\mu\nu} \in 1 \oplus 1 \]

\[ SO(3) \times SO(3) \]

\[ S_{\mu\nu} \in 0 \oplus 1 \oplus 2 \]

\text{symmetric (traceless) tensor}

\text{Spin representation:} \quad 2s + 1 = N_{(A_{\mu\nu})}

\[ A_{\mu} \Rightarrow \quad 2s + 1 = 3 \quad S = 1 \]

\[ SO(3) \]

\text{Scalar} \quad 2s + 1 = 1 \quad u(1)

\text{S = 0}

\[ A_{\mu\nu} \quad 2s + 1 = 6 \quad \Rightarrow \quad \frac{22}{22} \]

\[ SO(3) \times SO(3) \]

\[ 2s + 1 = 5 \quad S = 1 \]
\[ N_{\text{Comp}} = 2s_1+1 = 5 \quad (\text{spin } 2) \]
\[ 2s_1 = 3 \quad (\text{spin } 1) \]
\[ 2s_1 = 1 \quad (\text{spin } 0) \]

9 independent components

+1 for the addition of the trace

Recall again that we require massless representations, where we have only 2 DOF ( helicity ± s )

But we have NINE!

Let's take a lesson from history. How do we take a photo from \( A_{\mu} \)?

Yang-Mills

\[ A_{\mu} \rightarrow A_{\mu\nu} \underset{2 \times 2}{\text{diag}} \theta = \begin{bmatrix} \theta & \\ \theta & \end{bmatrix} \]

\[ \begin{align*}
\nabla \cdot A + \frac{\partial \phi}{\partial t} &= 0 \\
\nabla \cdot A &= -\frac{\partial \phi}{\partial t}
\end{align*} \]

4 DOF

- 2 conditions (gauge)

2 DOF \pm 1 polarization vectors

helicity

or \[ \partial_{\mu} A_{\mu} = 0 \]

or \[ \nabla \cdot A = 0 \] (no source)

+ \frac{\partial \phi}{\partial t} (no source)

2 conditions
So what gauge do we choose for the graviton
... a simple generalization

\[ h_{\mu \nu} \rightarrow h_{\mu \nu} - \left( \partial \phi - D \cdot \phi \right) \]

\[ \uparrow \quad \uparrow \]

Symmetric also symmetric

Given this, how do we construct an action consistent with everything we've said

- Symmetric trace free tensor (why?)
- Gauge invariance

With addition of "guess" quadratic in the fields
(think \( \phi^2 \) or \( \partial \phi \rightarrow \phi^2 \))

Will be derived (why?)

Consider all the terms we can construct

\[ 2 \partial \phi \cdot \partial \phi \]

\[ \partial \phi \cdot \partial \phi \quad \text{related} \quad \text{(extra credit for how)} \]

\[ \partial \mu \partial \nu \partial \phi \partial \phi \]

\[ \partial \mu \partial \nu \partial \phi \partial \phi \]
\[
S = \frac{1}{2} \int d^4x \left( -\partial \mu \partial \nu h_{\mu \nu} + 2 \partial \rho h_{\mu \nu} \delta^{\nu \rho} \right)
- \partial_{\rho} h_{\nu \alpha} \partial_{\alpha} h_{\rho \nu} + \partial_{\nu} h_{\rho \alpha} \partial_{\rho} h_{\alpha \nu} \right)
\]

Pauli - Fierz Action

Check w/ B-L equations:

\[
\frac{\partial J}{\partial (\partial_\mu h_{\nu \rho})} - \frac{\partial J}{\partial h_{\mu \nu}} \quad \text{true?}
\]

\[
- \partial_\sigma \partial_\rho h_{\mu \nu} \quad \Rightarrow \quad \Box^2 h_{\mu \nu}
\]

This works, so fixing gauge gives us GW (a spin 2 field now)

So pick the gauge which gives

\[
\partial^2 h_{\mu \nu} = 0 \quad \text{(Lorentz gauge - 4 conditions)}
\]

\[
\Box^2 h_{\mu \nu} \Rightarrow \Box^2 (h_{\mu \nu} + \delta_{\alpha \mu} \partial_\alpha \partial_\nu)
\]

\[
\Rightarrow \Box^2 \delta_{\mu \nu} \quad \text{(4 more conditions)}
\]
\[ g + 1 \] parameters
- 4 Lorentz
- 4 residual gauge

2 DoF

Good, it's what we expect. But does it reduce to the correct limit?

Propagator for a symmetric tensor field:

\[ \Sigma_{\mu\nu\rho} (k) = \frac{1}{2} \left( \Lambda_{\mu\nu} \Lambda_{\rho\sigma} + \Lambda_{\mu\rho} \Lambda_{\nu\sigma} - \Lambda_{\mu\sigma} \Lambda_{\nu\rho} \right) \left( \frac{-i}{k^2 - i\epsilon} \right) \]

Static limit $\rightarrow$ energy density $\rightarrow$ Potential Field

look at $D_{\alpha\sigma\sigma\sigma}$

\[ D_{\alpha\sigma\sigma\sigma} (k) = \frac{1}{2} \left( 1 + 1 - 1 \right) \left( \frac{-i}{k^2 - i\epsilon} \right) \]

\[ = \frac{-i}{2k^2} \]

Since $\phi$ scales which we know works.
But all of this is linear

GR is non-linear

In order to consider everything including an interaction term:

\[ \int \text{ansd} \]

we break our EM conservation theorem i.e.

\[ \partial_{\mu} T^{\mu}_{\nu} = 0 \quad \text{inside matter} \]

and therefore we need

\[ T^{\mu}_{\nu} \rightarrow T^{\mu}_{\nu} + \tau^{\mu}_{\nu} \]

but then we have

\[ \text{ansd} = \text{ansd} + \text{ansd} \]

where \( \text{ansd} \) is a function of \( \text{ansd} \)

which we have to add back into the action.
This gives us recursively higher order terms and represents the self interaction and the non-linearity of GR.