Thirring Model

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1 Introduction

The Thirring model is a completely soluble, covariant (1+1)-dimensional quantum field theory of a two-component Dirac spinor. We can write the generalized Thirring Lagrangian as,

$$\mathcal{L}_S = \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{i}{2}(\partial_\mu\bar{\psi})\gamma^\mu\psi - \frac{1}{2}(\partial_\mu A_\nu)B^{\mu\nu} + \frac{1}{2}A_\nu\partial_\mu B^{\mu\nu} - \frac{\mu^2}{2}A_\mu A^\mu + \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + gj^\mu A_\mu + \frac{\sigma}{2}j_\mu j^\mu.$$

I'll call this the Sommerfield Lagrangian for it is the more general case of the original Thirring Lagrangian,

$$\mathcal{L}_0 = i\bar{\psi}\gamma^\mu\partial_\mu\psi + \frac{\sigma}{2}j_\mu j^\mu$$

that Charles Sommerfield studied in the 1960's. In $\mathcal{L}_S$ we have the usual two-component Dirac spinor ψ but also a spin-1 field A_μ and a totally antisymmetric tensor field $B_{\mu\nu}$. This model could be relevant for it fits naturally with supersymmetry. In the Lagrangians I have implicitly defined the classical current,

$$j^\mu \equiv \bar{\psi}\gamma^\mu\psi.$$

Important: To attain total covariance of the theory (and for other reasons we will ecounter) we actually must more carefully define j^μ.
We will be using the "good-man's" metric $g^{00} = -g^{11} = -1$. The 2×2 Dirac matrices γ^μ obey the usual Clifford algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

We choose the representation via Pauli matrices,

$$\gamma^0 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We have the familiar

$$\gamma_5 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with the useful relation

$$\gamma^\mu \gamma^\nu = -g^{\mu\nu} + \epsilon^{\mu\nu}\gamma_5$$

where $\epsilon^{10} = -\epsilon^{01} = 1$ is the antisymmetric tensor. Using this antisymmetric tensor we can write

$$B = \frac{1}{2}\epsilon_{\mu\nu}B^{\mu\nu} \;\Rightarrow\; B_{\mu\nu} = -B\epsilon_{\mu\nu}$$

which explicitly shows that the field $B_{\mu\nu}$ has only one degree of freedom off-shell.

2 EoM

We can easily derive the equations of motions

$$-i\gamma^\mu \partial_\mu \psi = \gamma^\mu (gA_\mu + \sigma j_\mu)\psi$$

$$\partial_\nu B^{\mu\nu} = gj^\mu - \mu^2 A^\mu$$

and of course,

$$B_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

We also have the familiar field commutation relations

$$[A_1(x), B(x')] = i\delta(x - x')$$

and

$$\{\psi_\alpha(x), \psi_\beta(x')\} = \delta_{\alpha\beta}\delta(x - x').$$

3 The Action Principle

As with any quantum field theory, our goal is to compute vac-vac expectation values of time ordered products. Call such a time ordered product $(R)_+$. As usual the vac-vac amplitude with no source is,

$$\langle 0|0\rangle = \int \mathcal{D}A_\mu \mathcal{D}\,\psi \exp\left(i\int d^2x\,\mathcal{L}\right).$$

So varying the expectation value gives the general relation,

$$\delta\langle 0|(R) + |0\rangle = i\left\langle 0\left|\int d^2x\,(R\delta\mathcal{L})_+\right|0\right\rangle + \langle 0|\delta(R)_+|0\rangle.$$

The familiar procedure is to now introduce source terms which will allow us to gain insight on the vacuum. The external fields we introduce are ϕ_μ and J^μ. The the Lagrangian becomes,

$$\mathcal{L} \mapsto \mathcal{L} + \phi^\mu j_\mu + A_\mu J^\mu.$$

This modifies the equations of motion a bit,

$$-i\gamma^\mu \partial_\mu \psi = \gamma^\mu (gA^\mu + \sigma j_\mu + \phi_\mu)$$
$$\partial_\nu B^{\mu\nu} = gj^\mu + J^\mu - \mu^2 A^\mu.$$

From here on, we assume that the time-ordered products do not have explicit dependence on the external fields. So,

$$\frac{\delta}{\delta\phi_\mu(x)}\langle 0|(R)_+|0\rangle = i\langle 0|(j^\mu(x)R)_+|0\rangle$$
$$\frac{\delta}{\delta J^\mu(x)}\langle 0|(R)_+|0\rangle = i\langle 0|(A_\mu(x)R)_+|0\rangle.$$

But also,

$$\frac{\partial}{\partial g}\langle 0|(R)_+|0\rangle = i\left\langle 0\left|\int d^2x\,(Rj^\mu(x)A_\mu(x))_+\right|0\right\rangle$$
$$\frac{\partial}{\partial \sigma}\langle 0|(R)_+|0\rangle = \frac{i}{2}\left\langle 0\left|\int d^2x\,(Rj^\mu(x)j_\mu(x))_+\right|0\right\rangle.$$

This allows us to write out differential equationa for the vev's,

$$\frac{\partial}{\partial g}\langle 0|(R)_+|0\rangle_{e\sigma g} = -i\int d^2x\,\frac{\delta}{\delta\phi_\mu(x)}\frac{\delta}{\delta J^\mu(x)}\langle 0|(R)_+|0\rangle_{e\sigma g}$$
$$\frac{\partial}{\partial \sigma}\langle 0|(R)_+|0\rangle_{e\sigma g} = -\frac{i}{2}\int d^2x\,\frac{\delta}{\delta\phi_\mu(x)}\frac{\delta}{\delta \phi^\mu(x)}\langle 0|(R)_+|0\rangle_{e\sigma g}.$$

Where I have adopted the convention for the subscripts e, σ and g which means the external fields and couplings are "turned on". Simple integration yields,

$$\langle 0|(R)_+|0\rangle_{e\sigma g} = \exp\left[-ig\int d^2x\,\frac{\delta}{\delta\phi_\mu(x)}\frac{\delta}{\delta J^\mu(x)} - \frac{i}{2}\sigma\int d^2x\,\frac{\delta}{\delta\phi_\mu(x)}\frac{\delta}{\delta \phi^\mu(x)}\right]\times\langle 0|(R)_+|0\rangle_e.$$

The main point is that we now have an expression for the vev with coupling as a function of the vev without coupling, which is a much easier quantity to calculate Green's functions for. Specifically we have the definition of the n point fermionic Green's function

$$G^{(n)}_{c\sigma ge}(x_1,\ldots,x_n,x'_1,\ldots,x'_n) \equiv G^{(n)}_{c\sigma ge}(\bar{x},\bar{x}') = \frac{i^n\langle 0|(\psi(x_1)\cdots\psi(x_n)\bar{\psi}(x'_n)\cdots\bar{\psi}(x'_1))_+|0\rangle_{e\sigma g}}{\langle 0|0\rangle_{e\sigma g}}.$$

The subscripts maintain the same convention and the c just indicates that we're looking at the causal solutions. Now using our expression from above we can write,

$$G^{(n)}_{e\sigma g}(\bar{x},\bar{x}') = \frac{\left\{\exp\left[-ig\int d^2x\,\frac{\delta}{\delta\phi_\mu(x)}\frac{\delta}{\delta J^\mu(x)} - \frac{i}{2}\sigma\int d^2x\,\frac{\delta}{\delta\phi_\mu(x)}\frac{\delta}{\delta\phi^\mu(x)}\right]\right\}|G^{(n)}_{ce}(\bar{x},\bar{x}')\langle 0|0\rangle_e|}{\left\{\exp\left[-ig\int d^2x\,\frac{\delta}{\delta\phi_\mu(x)}\frac{\delta}{\delta J^\mu(x)} - \frac{i}{2}\sigma\int d^2x\,\frac{\delta}{\delta\phi_\mu(x)}\frac{\delta}{\delta\phi^\mu(x)}\right]\right\}\langle 0|0\rangle_e}.$$

Where we now explicitly have the form of the coupled Green's functions in terms of the uncoupled Green's functions.

4 Computing $G_{ec}^{(n)}(\bar{x}, \bar{x}')$

The n point Green's function can be written as the Slater determinant of 2 point Green's functions,

$$G_{ec}^{(n)}(\bar{x}, \bar{x}') = \begin{vmatrix} G_{ec}(x_1, x_1') & \cdots & G_{ec}(x_1, x_n') \\ \vdots & \ddots & \vdots \\ G_{ec}(x_n, x_1') & \cdots & G_{ec}(x_n, x_n') \end{vmatrix}.$$

Each of the two point functions satisfy the non-homogenous Dirac equation,

$$[-i\gamma^\mu \partial_\mu - \gamma^\mu \phi_\mu(x)] G_{ec}(x, x') = \delta(x - x').$$

Now turn ϕ off to get the regular Dirac equation,

$$-i\gamma^\mu \partial_\mu G_c(x, x') = \delta(x - x').$$

Recall the 2-d representation of the Dirac delta function,

$$\delta(x - x') = \int \frac{d^2 p}{(2\pi)^2} e^{ip^\mu (x-x')_\mu}.$$

So we have solution,

$$G_c(x, x') = \int \frac{d^2 p}{(2\pi)^2} \frac{-\gamma^\mu p_\mu}{p^2 - i\eta} e^{ip^\mu (x-x')_\mu} = \frac{1}{2\pi} \frac{\gamma^\mu (x - x')_\mu}{(x - x')^2 + i\eta}.$$

Thus,

$$G_c(x, x') = i\gamma^\mu \partial_\mu \tilde{\Delta}_c(x - x')$$

where we have introduced the propagator

$$\tilde{\Delta}_c(x - x') = -\frac{i}{4\pi} \log[(x - x')^2 + i\eta].$$

Now look at

$$[-i\gamma^\mu \partial_\mu - \gamma^\mu \phi_\mu(x)] \gamma^0 G_{ec}(x, x') = \delta(x - x') \gamma^0 \iff$$
$$(-g^{\mu 0} + \epsilon^{\mu 0} \gamma_5)(-i\partial_\mu - \phi_\mu) G_{ec}(x, x') = \gamma^0 \delta(x - x').$$

Take $\mu = 0$, then the LHS is just $(-i\partial_0 - \phi_0) G_{ec}$. Taking $\mu = 1$, the LHS is $(-i\gamma_5 \partial_1 - \gamma_5 \phi_1) G_{ec}$. So we can write,

$$-(i\partial_0 + i\gamma_5 \partial_1 + \phi_0 + \gamma_5 \phi_1) G_{ec}(x, x') = \gamma^0 \delta(x - x').$$

I claim that the external field solution is,

$$G_{ec}(x, x') = e^{i[F(x) - F(x')]} G_c(x - x')$$

where,

$$F(x) = -i \int d^2\xi \, G_c(x,\xi)\gamma^0[\phi_0(\xi) + \gamma_5\phi_1(\xi)].$$

First notice that,

$$G(x,\xi)\gamma^0[\phi_0(\xi) + \gamma_5\phi_1(\xi)] = \gamma^\mu\partial_\mu\tilde{\Delta}_c(x-\xi)\gamma^\nu\phi_\nu(\xi).$$

So we can write $F(x)$ more compactly as

$$F(x) = \gamma^\mu\partial_\mu \int d^2\xi \, \tilde{\Delta}_c(x-\xi)\gamma^\nu\phi_\nu(\xi).$$

Now,

$$\begin{aligned}
[-i\gamma^\mu\partial_\mu - \gamma^\mu\phi_\mu(x)]G_{ec}(x,x') &= \{[\gamma^\mu\partial_\mu F(x)G_c(x,x') + \delta(x-x')] - \gamma^\mu\phi_\mu(x)G_c(x,x')\}\exp[i(F-F')] \\
&= \left\{(\gamma^\mu\partial_\mu)^2\left[\int d^2\xi \; \tilde{\Delta}(x-\xi)\gamma^\nu\phi_\nu(\xi)\right] - \gamma^\mu\phi_\mu(x)\right\} G_c(x,x')\exp[i(F-F')] \\
&\quad + \; \delta(x-x') \\
&= \left\{\int d^2\xi \; \delta(x-\xi)\gamma^\nu\phi_\nu(\xi) - \gamma^\mu\phi_\mu(x)\right\} G_c(x,x')\exp[i(F-F')] \; + \delta(x-x') \\
&= \delta(x-x').
\end{aligned}$$

So it checks out. The determinant thus has the form,

$$\begin{aligned}
G^{(n)}_{ce}(\bar{x},\bar{x}') &= \sum_P (-1)^P \exp\left\{i\sum_i [F(x_i) - F(x'_{P(i)})]\right\} \prod_j G_c(x_j - x'_{P(j)}) \\
&= \sum_P (-1)^P \exp\left\{i\sum_i \int d^2\xi \; n^{(i)}_\mu(x_i, x'_{P(i)};\xi)\phi^\mu(\xi)\right\} \prod_j G_c(x_j - x'_{P(j)}).
\end{aligned}$$

Where,

$$\begin{aligned}
n^{(i)}_\mu(x_i, x'_{P(i)};\xi) &= \gamma^\nu\gamma_\mu\partial_\nu[\tilde{\Delta}(x-\xi) - \tilde{\Delta}(x'_{P(i)} - \xi)] \\
&= -(\partial_\mu + \gamma_5\bar{\partial}_\mu)[\tilde{\Delta}(x-\xi) - \tilde{\Delta}(x'_{P(i)} - \xi)].
\end{aligned}$$

5 (Re)Defining the Current

We look at the dependence of $\langle 0|0\rangle_e$ on the external fields. First recall,

$$\frac{\delta}{\delta\phi_\mu(x)}\langle 0|0\rangle_e = i\langle 0|j^\mu(x)|0\rangle.$$

Tentatively we wrote $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$. So referring to our expression for $G_c(x,x')$ and letting $\eta \to 0$ we have,

$$\begin{aligned}\frac{\langle 0|\bar{\psi}(x')\gamma^\mu\psi(x)|0\rangle_e}{\langle 0|0\rangle_e} &= -\frac{1}{2\pi}\mathrm{tr}_\alpha \exp[i(F(x)-F(x'))]\frac{\gamma^\nu(x-x')_\nu}{(x-x')^2} \\ &= -\frac{1}{2\pi}\mathrm{tr}_\alpha \exp[i(F(x)-F(x'))]\frac{\gamma^\nu\zeta_\nu}{a\zeta^2}.\end{aligned}$$

Where the trace is taken over all spinor indices and we have written $(x-x')^\mu \equiv a\zeta^\mu$. There is an obvious singularity as $x \to x'$. To remove this and also to preserve the charge symmetry $\psi \to \psi^\dagger$ we define,

$$j^\mu(x) = \frac{1}{2}\lim_{x'\to x}\left[\bar{\psi}(x')\gamma^\mu\psi(x) - \gamma^\mu\psi(x')\bar{\psi(x)}\right]$$

so that

$$\frac{i}{2}\lim_{x'\to x}\frac{\langle 0|\bar{\psi}(x')\gamma^\mu\psi(x) - \gamma^\mu\psi(x')\bar{\psi(x)}|0\rangle_e}{\langle 0|0\rangle_e} = -\frac{1}{4\pi}\lim_{x'\to x}\mathrm{tr}_\alpha\gamma^\mu\left\{\exp[iF(x)-iF(x')] - \exp[iF(x')-iF(x)]\right\} \times\frac{\gamma^\nu\zeta_\nu}{a\zeta^2}$$

About $x = x'$ we have to $\mathcal{O}(a^2)$,

$$\begin{aligned}\exp[iF(x)-iF(x')] - \exp[iF(x')-iF(x)] &= (1 + ia\partial_\lambda[F(x)-F(x')]\zeta^\lambda) - (1 - ia\partial_\lambda[F(x)-F(x')]\zeta^\lambda) \\ &= 2ia\partial_\lambda F(x)\zeta^\lambda\end{aligned}$$

Thus,

$$\begin{aligned}\frac{i}{2}\lim_{x'\to x}\frac{\langle 0|\bar{\psi}(x')\gamma^\mu\psi(x) - \gamma^\mu\psi(x')\bar{\psi(x)}|0\rangle_e}{\langle 0|0\rangle_e} &= -\frac{i}{2\pi}\frac{\zeta^\lambda\zeta^\nu}{\zeta^2}\mathrm{tr}_\alpha[\gamma^\mu\partial_\lambda F(x)\gamma_\nu] \\ &= -\frac{i}{\pi}\frac{\zeta^\lambda\zeta^\nu}{\zeta^2}[g^{\mu\sigma}g^\tau_\nu - \epsilon^{\mu\sigma}\epsilon^\tau_\nu]\partial_\sigma\partial_\lambda\int d^2\xi\ \tilde{\Delta}_c(x-\xi)\phi_\tau(\xi).\end{aligned}$$

There is one more subtlety we need to take into account in defining j^μ. Note that the above expression is not invariant with respect to the path by which we approach $x' \to x$ which takes the covariance out of the theory. To retain covariance we average over two directions of approach, namely ζ^μ and $\bar{\zeta}^\mu$. Our final result is (this time with spinor indices!),

$$\begin{aligned} j^\mu(x) &= \frac{1}{4}\gamma_{\alpha\beta} \lim_{a\to 0} \left[\bar{\psi}_\alpha\left(x-\frac{a}{2}\zeta\right)\psi_\beta\left(x+\frac{a}{2}\zeta\right)+\bar{\psi}_\alpha\left(x-\frac{a}{2}\bar{\zeta}\right)\psi_\beta\left(x+\frac{a}{2}\bar{\zeta}\right)\right. \\ &\left. -\psi_\beta\left(x-\frac{a}{2}\zeta\right)\bar{\psi}_\alpha\left(x+\frac{a}{2}\zeta\right)-\psi_\beta\left(x-\frac{a}{2}\bar{\zeta}\right)\bar{\psi}_\alpha\left(x+\frac{a}{2}\bar{\zeta}\right)\right]. \end{aligned}$$

You may ask at this point if all of this touching up with j^μ is even allowed. Lorentz invariance is obvious, but canonical commutation rules (as we shall see) will not be so simple. Also, the current as it is written is not invariant under the gauge transformation $\psi(x)\to\psi(x)+i\delta\Lambda(x)\psi(x)$ and $\bar{\psi}(x)\to\bar{\psi}(x)-i\delta\Lambda(x)\bar{\psi}(x)$, which will have to be considered. Now with this current,

$$\langle 0|0\rangle_e^{-1}\frac{\delta}{\delta\phi_\mu(x)}\langle 0|0\rangle_e = i\int d^2y\; v^{\mu\tau}(x-y)\phi_\tau(y)$$

where

$$v^{\mu\tau}(z)=\frac{1}{2\pi}(g^{\mu\sigma}g^{\nu\tau}-\epsilon^{\mu\sigma}\epsilon^{\nu\tau})\partial_\sigma\partial_\nu\tilde{\Delta}(z).$$

Integrating out,

$$\langle 0|0\rangle_e = \langle 0|0\rangle_J \exp\left[\frac{i}{2}\int d^2y\; d^2y'\; \phi_\mu(y)v^{\mu\nu}(y-y')\phi_\nu(y')\right].$$

Or more succinctly as,

$$\langle 0|0\rangle_e = \langle 0|0\rangle_J \exp[\phi v\phi].$$

Now look at,

$$\frac{\delta}{\delta J^\mu(x)}\langle 0|0\rangle = i\langle 0|A_\mu(x)|0\rangle_J.$$

Using equations of motion in the absence of ϕ_μ we have,

$$(-\Box+\mu^2)A_\mu(x)=J_\mu-\partial_\mu\partial_\nu A^\nu$$

and taking the inner product,

$$(-\Box+\mu^2)\langle 0|A_\mu(x)|0\rangle_J=\left[J_\mu(x)-\frac{1}{\mu^2}\partial_\mu\partial_\nu J^\nu(x)\right]\langle 0|0\rangle_J=\left[g_{\mu\nu}-\frac{1}{\mu^2}\partial_\mu\partial_\nu\right]J^\nu(x)\langle 0|0\rangle_J.$$

Integrating we find,

$$\langle 0|0\rangle_J=\exp\left[\frac{i}{2}\int d^2y\; d^2y'\; J^\mu(y)w_{\mu\nu}(y-y')J^\nu(y')\right] \text{"} = \text{"} \exp\left[\frac{i}{2}JwJ\right].$$

Where,

$$w_{\mu\nu}(z) = \left[g_{\mu\nu} - \frac{1}{\mu^2}\partial_\mu\partial_\nu\right]\Delta_c(\mu;z).$$

And finally

$$\Delta_c(\mu;z) = \frac{1}{(2\pi)^2}\int d^2p\,\frac{e^{ip^\mu z_\mu}}{p^2+\mu^2-i\eta}$$

which is the usual Klein-Gordon Green's function for a boson of mass μ in two dimensions.

6 Interlude: Completing the Square

We proceed to exponentiate quadratic functional derivatives of quadratic functionals in the fields. Let us write

$$\chi = \begin{pmatrix}\phi \\ J\end{pmatrix}$$

and

$$\mathcal{C} = -\begin{pmatrix}\sigma & g \\ g & 0\end{pmatrix} \qquad\qquad \mathcal{V} = \begin{pmatrix}v & 0 \\ 0 & w\end{pmatrix}.$$

Finally, let

$$\mathcal{N} = \mathcal{N}_\mu(x_i, x_{P(i)};\xi) = \begin{pmatrix}n_\mu(x_i, x_{P(i)};\xi) \\ 0\end{pmatrix}.$$

So the desired amplitude is,

$$\Omega \equiv \langle 0|0\rangle_{e\sigma g} = \exp\left[\frac{i}{2}\frac{\delta}{\delta\chi}\mathcal{C}\frac{\delta}{\delta\chi}\right](\exp[i\mathcal{N}\chi])(\exp[i/2\chi\mathcal{V}\chi]).$$

We can complete the square with the ansatz,

$$\chi' = \chi + \mathcal{N}\mathcal{V}^{-1}.$$

So that,

$$\Omega = \Omega' \exp\left[-\frac{i}{2}\mathcal{N}\mathcal{V}^{-1}\mathcal{N}\right]$$

where

$$\Omega' = \exp\left[\frac{i}{2}\frac{\delta}{\delta\chi'}\mathcal{C}\frac{\delta}{\delta\chi'}\right]\exp\left[\frac{i}{2}\chi'\mathcal{C}\chi'\right].$$

After differentiating, integrating and more massaging we get the final result,

$$\Omega = \Omega_0 \exp\left[-\frac{i}{2}\mathcal{N}\mathcal{C}(1+\mathcal{V}\mathcal{C})^{-1}\mathcal{N}\right]\exp[i\chi(1+\mathcal{V}\mathcal{C})^{-1}\mathcal{N}]\exp[i/2\chi(1+\mathcal{V}\mathcal{C})^{-1}\mathcal{V}\chi].$$

And where the amplitude of the free theory $\Omega_0 \equiv \langle 0|0\rangle_{\sigma g}$ (with no external fields) is

$$\Omega_0 = \exp\left[-\frac{1}{2}\,\mathrm{tr}\,\log(1+\mathcal{V}\mathcal{C})\right].$$

7 Properties of the Solution

Converting the language from the last section we have,

$$\langle 0|0\rangle_{e\sigma g} = \Omega_0 \exp\left[\frac{i}{2}\phi V\phi + i\phi UJ + \frac{i}{2}JWJ\right]$$

which is in our short notation

$$fXg \equiv \int d^2\xi\, d^2\xi'\ f_\mu(\xi)X^{\mu\nu}(\xi-\xi')g_\nu(\xi').$$

And where (I list here for reference only. It is good to see the structure of these functions as they come from the act of Gaussian integration.)

$$V_{\mu\nu}(z) = \frac{1}{2\pi}\left[\frac{-2\partial_\mu\partial_\nu}{1-(\alpha+\lambda)^2}\tilde{\Delta}_c(z) + \frac{\lambda(\mu' g_{\mu\nu} - \partial_\mu\partial_\nu)}{(1+\alpha)(1+\alpha+\lambda)}\Delta_c(\mu';z) - \frac{g_{\mu\nu}}{1+\alpha}\delta(z)\right]$$

$$U_{\mu\nu}(z) = \frac{g}{2\pi\mu^2}\left[\frac{-2\partial_\mu\partial_\nu}{1-(\alpha+\lambda)^2}\tilde{\Delta}_c(z) - \frac{\mu' g_{\mu\nu} - \partial_\mu\partial_\nu}{1+\alpha+\lambda}\Delta_c(\mu';z)\right]$$

$$W_{\mu\nu}(z) = \frac{1}{\mu^2}\left[\frac{-2\lambda\partial_\mu\partial_\nu}{1-(\alpha+\lambda)^2}\tilde{\Delta}_c(z) + \frac{(1+\alpha)(\mu' g_{\mu\nu} - \partial_\mu\partial_\nu)}{1+\alpha+\lambda}\Delta_c(\mu';z)\right].$$

The coupling constants are adjusted,

$$\alpha = \frac{\sigma}{2\pi}$$

$$\lambda = \frac{g^2}{2\pi\mu^2}$$

and we have an adjusted mass for the Bosonic Green's function,

$$\mu'^2 = \frac{1+\alpha+\lambda}{1+\alpha}\,\mu^2.$$

The free theory vac-vac amplitude is computed as,

$$\langle 0|0\rangle_{\sigma g} = \exp\left\{-\frac{1}{2}\int d^2x\ \left[\delta^{(2)}(0)\log[(1-\alpha-\lambda)(1+\alpha)] - \frac{1}{2}\int_{\mu^2}^{\mu'^2} dr^2\ \Delta_c(r;0)\right]\right\}.$$

This gives a trivial result if you look at it closely. We will reexamine this when look at a modified Lagrangian and it so happens that the Delta functions cancel and we are left

with a non-trivial finite amplitude.
The two-point fermionic Green's function with sources off turns out to be,

$$G_{\sigma gc}(x,x') = \Bigg[\exp 2\pi i \Bigg\{ \frac{2(\alpha+\lambda)^2}{1-(\alpha+\lambda)^2}[\tilde{\Delta}_c(0)-\tilde{\Delta}_c(x-x')] + \frac{\lambda}{(1+\alpha)(1+\alpha+\lambda)}[\Delta_c(\mu';0)-\Delta_c(\mu';x-x')]\Bigg\}\Bigg] G_c(x-x')$$

As this two point function stand it is ultraviolet divergent for $\tilde{\Delta}_c(0)$ and $\Delta_c(\mu';0)$ diverge at high energies. There is also infrared difficulties with the fermionic Green's function but can be avoided by defining asymptotic fermionic states. We must supply an ultraviolet cutoff in order to further make sense of the limiting process of defining the current.

8 Ultraviolet Cutoff

9 Many-Fermion Green's Function

10 Bosonic Operators

11 Commutation Relations

12 Gauge Symmetries

13 Making Sense of the Current

14 New Lagrangian and Correcting the Vac-Vac Amplitude