Introduction to String Theory

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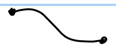
① The Classical String

$x^\mu(\tau)$ Point Particle → Quantum Mechanics → (Local) Quantum Field Theory

$\{,\}_{PB} \to \frac{1}{i}[,]$

need for creation and anhilation of particles (Special Relativity)



$X^\mu(\tau,\sigma)$ One dimensional extended object (String) → String Theory

$\{,\}_{PB} \to \frac{1}{i}[,]$

(Relativistic theory with creation and anhilation of particles)

Relativistic Point Particle

length of the world-line.

$$S_0 = -m\int ds \quad , \quad ds^2 = \eta_{\mu\nu}\,dx^\mu dx^\nu = \eta_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu (d\tau)^2$$

$$S_0 = -m\int \sqrt{\eta_{\mu\nu}\,\dot{x}^\mu(\tau)\,\dot{x}^\nu(\tau)}\; d\tau$$

$\delta S = 0$ (equations of motion)

$$\Longrightarrow \frac{d^2x^\mu}{d\tau^2} = 0$$

Notice that the action So is reparametrization invariant:

$$\tau \to \tau'(\tau) \qquad S_0 \to -m\int d\tau' \frac{d\tau}{d\tau'}\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\tau'}\frac{d\tau'}{d\tau}\frac{dx^\nu}{d\tau'}\frac{d\tau'}{d\tau}}$$

$$= -m\int d\tau' \cancel{\frac{d\tau}{d\tau'}}\,\cancel{\frac{d\tau'}{d\tau}}\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\tau'}\frac{dx^\nu}{d\tau'}} = -m\int d\tau' \sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\tau'}\frac{dx^\nu}{d\tau'}}$$

This action is awkward for two reasons:

- The square root is mathematically difficult for quantization
- Not useful for massless ($m=0$) particles

$$\int Dx\, e^{i\int d\tau\sqrt{\dot{x}^2}}$$

$$\underbrace{\int Dx\, e^{i\int d\tau\, \dot{x}^2}}_{\text{Gaussian}}$$

We can construct a different action equivalent at the classical level (i.e: that gives the same equations of motion)

$$\tilde{S}_0 = \frac{1}{2}\int d\tau\left[e^{-1}(\tau)\,\dot{x}^2 - m^2 e(\tau)\right] \quad , \quad e(\tau): \text{auxiliary field}$$

$$\delta\tilde{S}_{0\,(\delta e)} = \frac{1}{2}\int d\tau\left[-e^{-2}\dot{x}^2\,\delta e - m^2\,\delta e\right] = -\frac{1}{2}\int d\tau\left[e^{-2}\dot{x}^2 + m^2\right]\delta e$$

$$\Longrightarrow \quad \dot{x}^2 + e^2(\tau)\,m^2 = 0 \qquad e^2 = \frac{-\dot{x}^2}{m^2}$$

Plugging this back into $\tilde{S}_0$

$$\tilde{S}_0 = \frac{1}{2}\int d\tau\left[-\frac{m}{\sqrt{\dot{x}^2}}\,\dot{x}^2 - m^{\cancel{2}}\,\frac{\sqrt{-\dot{x}^2}}{\cancel{m}}\right] = -m\int d\tau\sqrt{\dot{x}^2}$$

$$\delta\tilde{S}_{0\,(\delta x)} = 0 \quad\rightarrow\quad \frac{d^2x^\mu}{d\tau^2} = 0$$

$\tilde{S}_0$ is also re-paremetrization invariant IF $e(\tau)$ is a einbein (ie: 1+0 space-time metric) this means:

$$e'(\tau') \neq e(\tau) \quad \rightarrow \text{not a scalar field}$$

metric on the world-line $\longleftarrow$ $$e'(\tau') = \frac{d\tau}{d\tau'}\,e(\tau) \qquad \left(g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\,g_{\alpha\beta}(x)\right)$$

But $x'^\mu(\tau') = x^\mu(\tau) \rightarrow$ Scalar field

$$\tilde{S}_0 = \frac{1}{2}\int d\tau\left[e^{-1}\dot{x}^2 - m^2 e\right] \qquad \tau \rightarrow \tau' = \tau - \varepsilon(\tau)$$

$$x'^\mu(\tau-\varepsilon) = x^\mu(\tau)$$

$$x'^\mu(\tau) - \varepsilon\frac{dx'^\mu}{d\tau} = x^\mu(\tau) \quad\rightarrow\quad \delta x^\mu(\tau) \equiv x'^\mu(\tau) - x^\mu(\tau) = \varepsilon\frac{dx'^\mu}{d\tau} + \theta(\varepsilon^2)$$

$$\delta x^\mu(\tau) = \varepsilon\,\dot{x}^\mu + \theta(\varepsilon^2)$$

$$\frac{d\tau'}{d\tau}\,e'(\tau') = e(\tau) \quad\rightarrow\quad (1-\dot{\varepsilon})(e'(\tau) - \varepsilon\,\dot{e}'(\tau)) = e(\tau)$$

$$e'(\tau) - \varepsilon\,\dot{e}'(\tau) - \dot{\varepsilon}\,e'(\tau) + \theta(\varepsilon^2) = e(\tau)$$

$$\delta e(\tau) = \frac{d}{d\tau}(\varepsilon e) + o(\varepsilon^2)$$

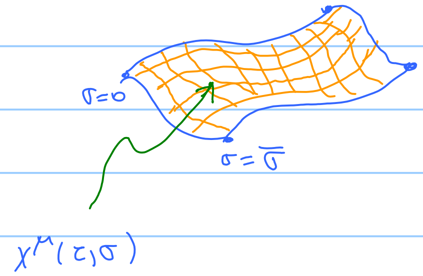
$$\delta \hat{\tilde{S}}_0 = \frac{1}{2}\int d\tau \left[e^{-1} 2\dot{x}_\mu \delta x^\mu - e^{-2}\delta e \dot{x}^2 - m^2 \delta e \right]$$

After some algebra

$$\delta \tilde{S}_0 = \frac{1}{2}\int d\tau \frac{d}{d\tau}\left[\frac{\varepsilon}{e}\dot{x}^2 \right] = \text{Boundary term} = 0$$

$\therefore$ $\tilde{S}_0$ is reparametrization invariant.

Relativistic String



Try:

Action = Area of the world-sheet

$$S_{NG} = -T\int d\tau d\sigma \sqrt{(\dot{x}\cdot x')^2 - \dot{x}^2 x'^2} \qquad \text{Nambu-Goto action}$$

$$\dot{x}^\mu \equiv \frac{dx^\mu}{d\tau} \qquad x'^\mu \equiv \frac{dx^\mu}{d\sigma}$$

Again, this is action is not well suited for quantization.
Alternative : Find another action with the same eqs. of motion but without the $\sqrt{}$. Need of auxiliary field

$$S_{Pol} = -\frac{T}{2}\int d\tau d\sigma \sqrt{-h}\, h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \qquad \partial_\alpha \equiv \frac{\partial}{\partial\sigma^\alpha}$$

$h_{\alpha\beta}(\tau\sigma)$: auxiliary field → metric on the world-sheet. $\sigma^\alpha = (\tau,\sigma) \to (\alpha=1,2)$

$h_{\alpha\beta}$ in fact has the interpretation of a world-sheet metric i.e.:

$$h'_{\alpha\beta}(\tau',\sigma') = \frac{\partial\sigma^{\gamma}}{\partial\sigma'^{\alpha}}\frac{\partial\sigma^{\delta}}{\partial\sigma'^{\beta}}\, h_{\gamma\delta}(\tau,\sigma) \qquad (1+1 \text{ metric})$$

Equations of motion:

Note that the Polyakov action looks like the action for d scalar fields $X^{\mu}(\tau,\sigma)$ $(\mu = 0,1,\dots,d-1)$ as a two-dimensional field theory where the manifold they're defined on is the world-sheet

E.o.m's :

$$\begin{array}{cc} \dfrac{\delta S}{\delta h^{\alpha\beta}} = 0 & \dfrac{\delta S}{\delta X^{\mu}} = 0 \\ \downarrow & \downarrow \\ T_{\alpha\beta} = 0 & \dfrac{1}{\sqrt{h}}\partial_{\alpha}\left(\sqrt{h}\, h^{\alpha\beta}\partial_{\beta}X^{\mu}\right) = 0 \end{array}$$

↳ Wave equation in a background

Symmetries of the Polyakov action:

1) Global symmetries:

- Poincaré :

$$X^{\mu}(\tau,\sigma) \to X'^{\mu}(\tau,\sigma) = \epsilon^{\mu}{}_{\nu}X^{\nu}(\tau,\sigma) + a^{\mu}$$

$$(\epsilon_{\mu\nu} = -\epsilon_{\nu\mu})$$

2) Local symmetries:

- Reparametrization Invariance: Change world sheet coordinates.

$$\sigma^\alpha \longrightarrow \sigma'^\alpha = \sigma'^\alpha(\sigma^\alpha) \quad \alpha = 1, 2.$$, in other words

$$\tau \longrightarrow \tau' = \tau'(\tau,\sigma)$$
$$\sigma \longrightarrow \sigma' = \sigma'(\tau,\sigma)$$

This has the same meaning of general coordinate invariance of General Relativity.

Under these, $X^\mu(\tau,\sigma)$ and $h_{\alpha\beta}(\tau,\sigma)$ transform as

$$X'^\mu(\tau',\sigma') = X^\mu(\tau,\sigma)$$
$$h'_{\alpha\beta}(\tau',\sigma') = \frac{\partial\sigma^\gamma}{\partial\sigma'^\alpha}\frac{\partial\sigma^\delta}{\partial\sigma'^\beta} h_{\gamma\delta}(\tau,\sigma)$$

- Weyl Rescalings: No change of world-sheet coordinates involved. Only we re-scale the metric at each world-sheet point

That is called conformal transformation.

$$h_{\alpha\beta}(\tau,\sigma) \longrightarrow h'_{\alpha\beta}(\tau,\sigma) = \Omega(\tau,\sigma)\, h_{\alpha\beta}(\tau,\sigma)$$

Why are these symmetries important? For many reasons. For example, when quantizing the theory, we need to take into account these extra degrees of freedom and find the correct Haar measure to get a sensical path integral (Fadeev-Popov ghosts will be indeed introduced)

We therefore need to fix the gauge to be able to find "physical" answers (we mean gauge independent answers to be more precise)

Gauge Fixing

$$h_{\alpha\beta} = \begin{pmatrix} h_{00}(\tau,\sigma) & h_{01}(\tau,\sigma) \\ h_{01}(\tau,\sigma) & h_{11}(\tau,\sigma) \end{pmatrix} \rightarrow 3 \text{ local functions}$$

thus, we need 3 local conditions (conditions that are applied at every single world-sheet point (τ,σ)) to completely fix the form of $h_{\alpha\beta}$.

We cannot use the Poincaré symmetry to fix the gauge because it's a global one. We need local symmetries instead:

$$\text{Re-parametrization} : (\tau,\sigma) \rightarrow \left.\begin{matrix} \tau' = f(\tau,\sigma) \\ \sigma' = g(\tau,\sigma) \end{matrix}\right\} \begin{matrix} 2 \text{ } \underline{\text{local}} \\ \text{parameters} \\ (f \text{ and } g) \end{matrix}$$

$$\text{Weyl re-scalings} : h_{\alpha\beta}(\tau,\sigma) \rightarrow \underbrace{\Omega(\tau,\sigma)}_{1 \text{ local parameter}} h_{\alpha\beta}(\tau,\sigma)$$

$2+1 = 3 \longrightarrow$ we can fix the components of $h_{\alpha\beta}(\tau,\sigma)$ completely for each world-sheet point (τ,σ)

A very convenient one is

$$\underbrace{h_{\alpha\beta}(\tau,\sigma) = \Lambda(\tau,\sigma)\,\eta_{\alpha\beta}}_{\text{This is called Conformal gauge. *}}$$

$$\therefore\; ds^2 = -d\tau^2 + d\sigma^2$$

In this gauge, the Polyakov action take the form:

$$S_{Pol} = \frac{T}{2}\int d\tau\, d\sigma\,(\dot{X}^2 - X'^2)$$

Let's derive the equations of motion directly from here:

$$\delta S_{Pol} = T\int d\tau\, d\sigma\, \delta X^\mu(\partial_\sigma^2 - \partial_\tau^2)X_\mu - T\int_{\tau_1}^{\tau_2} d\tau\, X'_\mu\, \delta X^\mu \Big|_{\sigma=0}^{\sigma=\bar{\sigma}}$$

Key point:

Note that the boundary term is not the typical boundary term we have for point particles which is to decide where the paths are static or not at $\tau = \tau_1$ or $\tau = \tau_2$ (initial conditions). Here we have boundary conditions for the end-points of the string!

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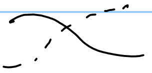
* This is possible as long as the world-sheet has no topological obstructions. This is valid for manifolds with $\chi = 0$ (Euler character)

To get $\delta S_{pol} = 0$ we need the e.o.m.

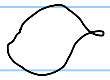
$$(-\partial_\tau^2 + \partial_\sigma^2) X^\mu(\tau,\sigma) = 0$$ ← wave eq. on flat 2D manifold (as expected)

and

$$X'_\mu \, \delta X^\mu \Big|_{\sigma=0}^{\sigma=\bar\sigma} = X'_\mu(\tau,\sigma=\bar\sigma)\,\delta X^\mu(\tau,\sigma=\bar\sigma) - X'_\mu(\tau,0)\,\delta X^\mu(\tau,0) = 0$$

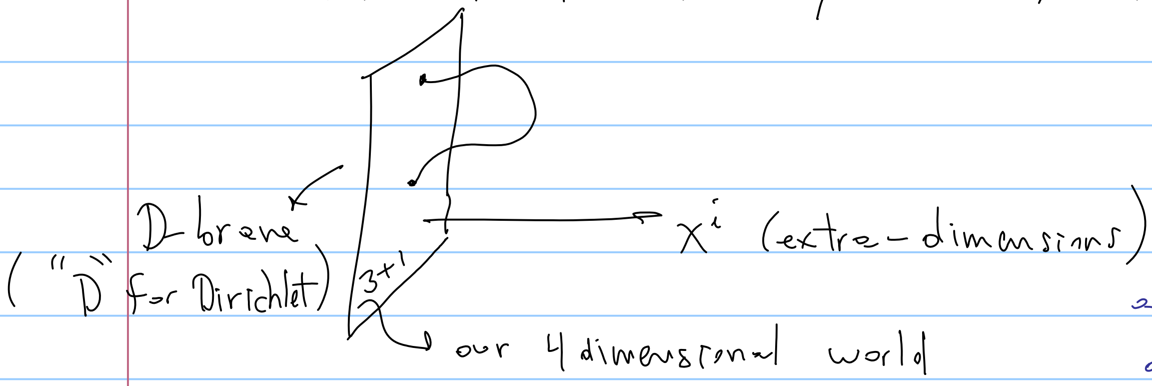
1) $X'_\mu(\bar\sigma) = X'_\mu(\sigma) = 0$  ∴ Neumann (Open String)

2) $\delta X^\mu(\tau,\sigma=0) = \delta X^\mu(\tau,\sigma=\bar\sigma) = 0$  ∴ Dirichlet
(fixed end points) (Open String)

3) $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\bar\sigma)$ →  ∴ Closed string

* We can have a fourth option: fix Neumann for some components of X^μ and Dirichlet for the rest: For example:

Neumann for X^μ $\mu = 0,1,2,3$.
Dirichlet for X^μ $\mu = 4,6,\ldots,d$



Note that the end-points of the string are free to move in all the four dimensions of 'our world'.

Solutions to the equations of motion

Depending on the boundary conditions imposed, the solutions are slightly different:

$$(-\partial_\tau^2 + \partial_\sigma^2)\, X^\mu(\tau,\sigma) = 0$$

- Closed string:

$$X^\mu(\tau,\sigma) = \underbrace{X_R^\mu(\tau-\sigma)}_{\text{right-mover}} + \underbrace{X_L^\mu(\tau+\sigma)}_{\text{left-mover}}$$

We can express these functions in a plane wave decomposition

$$X_R^\mu(\tau-\sigma) = \frac{1}{2} x^\mu + \frac{1}{4\pi T}(\tau-\sigma)p^\mu + \frac{i}{\sqrt{4\pi T}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in(\tau-\sigma)}$$

$$X_R^\mu(\tau-\sigma) = \frac{1}{2} x^\mu + \frac{1}{4\pi T}(\tau+\sigma)p^\mu + \frac{i}{\sqrt{4\pi T}} \sum_{n\neq 0} \frac{1}{n} \bar{\alpha}_n^\mu e^{-in(\tau+\sigma)}$$

Note the α_n and $\bar{\alpha}_n$ do not need to be related. They are just all different expansion coefficients.

However, the requirement that $X^\mu(\tau,\sigma)$ is a real function implies:

- x^μ, p^μ are real
- $\alpha^\mu_{-n} = (\alpha_n^\mu)^*$ & $\bar{\alpha}^\mu_{-n} = (\bar{\alpha}_n^\mu)^*$

(to be continued)