Multi-mode quantum noise model for advanced gravitational wave detectors

Project Report

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Contents

1 Abstract

We calculate the power spectral density of the light field incident on a photodiode for several situations involving multiple modes of coherent light and squeezed light. We compare these analytical results to those of FINESSE simulations of the same situation in order to verify the accuracy of the new multi-mode squeezing package of the program. We aim to update FINESSE as needed until it can match analytical results for these simulations; at this point the multi-mode squeezing package can be released for public use.

First, we consider purely fundamental Hermitian-Gauss mode (HG_{00}) coherent light and (HG_{00}) amplitude-squeezed light in a set-up lacking a filter cavity. We then contemplate the same situation, but consider a higher-order mode (HG_{01}) in additional to the fundamental mode for both the coherent and squeezed optical fields. We squeeze only the fundamental mode of the light leaving the squeezed source. Next, we add a filter cavity to the single-mode situation, which causes squeezing to be frequency-dependent. Finally, we consider two modes (HG_{00} and HG_{01}) for the coherent and squeezed optical fields in a set-up including a filter cavity. We compare the power spectral density results for each of these situations with the results of FINESSE simulations for the same situations in order to verify the accuracy of the multi-mode squeezing package of the program.

2 Introduction to FINESSE and the Use of Squeezed Light in Gravitational Wave Detectors

FINESSE (Frequency domain INterfErometer Simulation SoftwarE) is a program created by Dr. Andreas Freise used to design and debug laser interferometers. It has been employed for years by the optics groups of gravitational wave (GW) detectors and is consistently being updated to contribute to the latest developments in the field. FINESSE is not currently equipped to simulate situations involving multi-mode squeezed light, although a new package has been developed that will allow it to do so. My work this summer was focused on verifying the accuracy of this new package before it is released for public use.

Squeezed light, which was first demonstrated in 1985, is the result of coherent light being focused into a non-linear crystal. This which polarizes the crystal and causes an effect known as parametric down-conversion (PCD) in which one photon creates two daughter photons of about twice the initial wavelength. This process dampens the quantum noise of the optic field as depicted below:

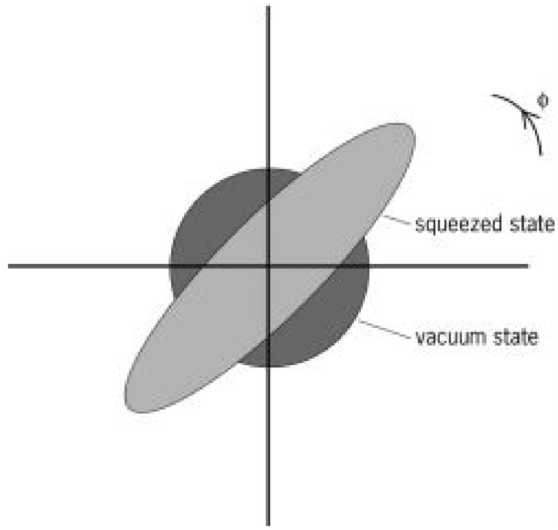

Figure 1: A vacuum state is squeezed at angle phi. The vertical axis represents amplitude and the horizontal axis represents phase.

Light can be squeezed at a variety of angles depending on the noise-dampening desired for a given situation. For example, at low frequencies, most of the quantum noise in a GW detector comes from the amplitude quadrature of the optical field and is called radiation-pressure noise. Therefore, light should be amplitude-squeezed at low frequencies in order to lower the quantum noise as much as possible. At high frequencies, the phase quadrature causes most of the noise, which is called shot noise, so light should be phase-squeezed at high frequencies. At intermediate frequencies, a squeezing angle that squeezes both the phase and amplitude noise should be used. In the first portion of this report (sections 3 and 4), the squeezing angle is set to be amplitude-squeezing for all frequencies, while in the latter portion (sections 5 and 6), the squeezing will be frequency-dependent because of the addition of a filter cavity.

The apparatus in which this squeezing process takes place is called a squeezing source. For the purposes of this project, only the fundamental Hermitian-Gauss mode (HG_{00}) of the light sent through the squeezing source is squeezed, even if higher-order modes (HG_{01}, for example) are present. This means the non-linear crystal used in the squeezing source is structured in such a way that it does not change the quantum noise of light in high-order modes. Therefore, although the light produced in the squeezing source may have multiple modes, only the fundamental mode is truly squeezed light (this will become mathematically evident when calculating power spectral density, as squeezing affects the spectral densities of the phase and amplitude quadratures of the optical field). The higher-order modes of the light from the squeezing source have the same quadrature spectral densities as coherent light. However, this does not mean that the higher-order modes of the squeezed light are equivalent to those of any other coherent light, as they may not be from the same source and thus may not have the same amplitude. In fact, squeezed light contains so few photons that we consider its amplitude to be negligible in comparison to that of the coherent light. In this project, the squeezed source and laser source are separate, so the squeezed light higher-order modes are different from those of the coherent light.

In the most recent update to LIGO, the advanced GW detector was equipped to use squeezed light to reduce quantum noise. Therefore, software that can accurately model squeezed light in multi-mode scenarios is necessary to allow for further advancements to be made in its use of squeezed light. FINESSE has been instrumental in many past improvements to LIGO and this new package will allow it to continue to be so in the study of the benefits of squeezed light.

3 Single Mode Coherent Light and Squeezed Light

3.1 Calculating Power

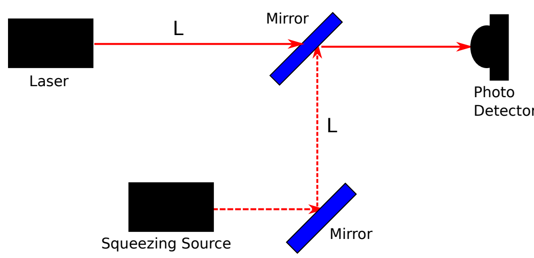

Figure 2: HG_{00} squeezed light is combined with HG_{00} coherent light before reaching a photo detector.

We can describe coherent light, fully in the fundamental Hermitian-Gauss mode (HG_{00}), in the following way immediately after it leaves the laser source:

$$\hat{E}_L(t, z) = A\cos(\Delta(t)) + \hat{a}_1(t)\cos(\Delta(t)) + \hat{a}_2(t)\sin(\Delta(t)) \tag{1}$$

where $\Delta = \omega_0 t - kz$.
In the above equation, A is the amplitude of the laser, k is its wave number, t is time at which the field is measured, z is position at which it is measured, $\hat{a}_1(t)$ is the amplitude quadrature of the wave, and $\hat{a}_2(t)$ is the phase quadrature of the wave.

We can describe squeezed light, also fully in the fundamental mode HG_{00}, in the following way immediately after it leaves the squeezing source:

$$\hat{E}_{SL} = \hat{a}_{1S}(t)\cos(\Delta(t) - \theta) + \hat{a}_{2S}(t)\sin(\Delta(t) - \theta) \tag{2}$$

where $\hat{a}_{1S}(t)$ is the amplitude quadrature of the squeezed light, $\hat{a}_{2S}(t)$ is its phase quadrature, and θ is an arbitrary phase difference between the squeezed light and the coherent light that can be set to any desired value. The squeezed light has no amplitude term A because it is comprised of so few photons that its amplitude is negligible in comparison to that of the coherent laser light.

We assume that the first mirror is a negligible distance away from the squeezing source so there is no phase change as the field propagates from the source to the mirror. We also assume the bottom mirror has a reflectivity of 1 so all of the squeezed light is reflected upon reaching it. Thus the squeezed light equation $\hat{E}_{SL}$ is not affected by that mirror.

The squeezed light then travels a distance L to the top mirror. The field E_{SL} undergoes a phase change as it propagates that distance. Time has passed as it propagated so the optical field is now a function of t'. Therefore, just before reaching the top mirror:

$$\begin{aligned}
\hat{E}_{SL} &= \hat{a}_{1S}(t')\cos(\Delta(t') - \theta) + \hat{a}_{2S}(t')\sin(\Delta(t') - \theta) \\
&= \hat{a}_{1S}(t')\cos(\Delta(t'))\cos(\theta) + \hat{a}_{1S}(t')\sin(\Delta(t'))\sin(\theta) + \hat{a}_{2S}(t')\sin(\Delta(t'))\cos(\theta) - \hat{a}_{2S}(t')\cos(\Delta(t'))\sin(\theta) \\
&= [\hat{a}_{1S}(t')\cos(\theta) - \hat{a}_{2S}(t')\sin(\theta)]\cos(\Delta(t')) + [\hat{a}_{1S}(t')\sin(\theta) + \hat{a}_{2S}(t')\cos(\theta)]\sin(\Delta(t'))
\end{aligned} \tag{3}$$

The coefficients of this equation can be written in matrix form.

$$\begin{pmatrix} \hat{b}_{1S}(t') \\ \hat{b}_{2S}(t') \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \hat{a}_{1S}(t') \\ \hat{a}_{2S}(t') \end{pmatrix} \tag{4}$$

where $\hat{b}_{1S}(t')$ is the coefficient of the $\cos(\Delta(t'))$ term and $\hat{b}_{2S}(t')$ is the coefficient of the $\sin(\Delta(t'))$ term.

We assume the coherent laser light also travels a distance L before reaching the top mirror so the only phase difference it has from the squeezed light is θ. We also assume the top mirror has a very high reflectivity and a very low transmissivity so very little coherent light (which has a high amplitude) is transmitted while almost all of the squeezed light is reflected. This allows the noise-reducing effects of the squeezed light to be maximized while also providing enough amplitude from the coherent light for the field to be recognized by the photo detector. The squeezed light (reflected by the mirror) and the coherent light (transmitted through the mirror) combine to make $\hat{E}_{SLS}$, which has the following coefficients for its $\cos(\Delta(t'))$ and $\sin(\Delta(t'))$ terms:

$$\begin{pmatrix} \hat{b}_1(t') \\ \hat{b}_2(t') \end{pmatrix} = r_2 \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \hat{a}_{1S}(t') \\ \hat{a}_{2S}(t') \end{pmatrix} + t_2 \begin{pmatrix} \hat{a}_1(t') \\ \hat{a}_2(t') \end{pmatrix} + t_2 \begin{pmatrix} A \\ 0 \end{pmatrix} \tag{5}$$

The reflection coefficient of the mirror is r_2 and the transmission coefficient is t_2. We assume the distance between the mirror and the photo diode is negligible so $\hat{E}_{LSL}$ does not undergo a phase change while propagating towards the photo diode.

The power observed by the photo diode is $P = \hbar\omega_0 \hat{E}_{LSL}^2$. Ignoring higher order terms that occur when quadratures are multiplied by one another leaves terms from the quadratures multiplied by the amplitude and the constant term from the amplitude squared.

$$P(t,z) = \hbar\omega_0\{2r_2t_2A[\cos(\theta)\hat{a}_{1S}(t') - \sin(\theta)\hat{a}_{2S}(t')]\cos^2(\Delta(t')) + 2t_2^2A\hat{a}_1(t')\cos^2(\Delta(t')) + t_2^2A^2\cos^2(\Delta(t'))\} \tag{6}$$

Next we apply double angle formulas to the trigonometric squared terms and remove the resulting fast-oscillating terms of the form $\cos(2\omega_0 t)$ or $\sin(2\omega_0 t)$. These pieces can be removed because the photo diode can only resolve frequencies up to around 10 MHz and ω_0 is in the THz order of magnitude. In addition, power is defined as the average energy per period of the oscillating electromagnetic field, so the oscillating terms can be ignored as their time-averages are zero. This yields the power:

$$P(t,z) = \hbar\omega_0 t_2 A\{r_2[\cos(\theta)\hat{a}_{1S}(t') - \sin(\theta)\hat{a}_{2S}(t')] + t_2\hat{a}_1(t') + \frac{t_2 A}{2}\} \tag{7}$$

Removing the constant amplitude term yields the power contributed by the quadratures (the fluctuating power):

$$P(t,z) = \hbar\omega_0 t_2 A\{r_2[\cos(\theta)\hat{a}_{1S}(t') - \sin(\theta)\hat{a}_{2S}(t')] + t_2\hat{a}_1(t')\} \tag{8}$$

3.2 Calculating Power Spectral Density

Power spectral density is defined as

$$\frac{1}{2}S(\Omega)_{PP}\delta(\Omega - \Omega') \equiv \frac{1}{2\pi}\langle 0|P(\Omega)P(\Omega')^\dagger + P(\Omega')P(\Omega)^\dagger|0\rangle \tag{9}$$

Fourier transforming $P(t,z)$ yields $P(\Omega,z)$:

$$P(\Omega,z) = \hbar\omega_0 t_2 A\{r_2[\cos(\theta)\hat{a}_{1S}(\Omega) - \sin(\theta)\hat{a}_{2S}(\Omega)] + t_2\hat{a}_1(\Omega)\} \tag{10}$$

Substituting this power into the power spectral density equation:

$$\frac{1}{2}S(\Omega)_{PP}\delta(\Omega-\Omega')$$
$$=(\hbar\omega_0 t_2 A)^2\frac{1}{2\pi}\langle 0|\{r_2[\cos(\theta)\hat{a}_{1S}(\Omega)-\sin(\theta)\hat{a}_{2S}(\Omega)]+t_2\hat{a}_1(\Omega)\}\{r_2[\cos(\theta)\hat{a}^\dagger_{1S}(\Omega')-\sin(\theta)\hat{a}^\dagger_{2S}(\Omega')]+t_2\hat{a}^\dagger_1(\Omega')\}$$
$$+\{r_2[\cos(\theta)\hat{a}_{1S}(\Omega')-\sin(\theta)\hat{a}_{2S}(\Omega')]+t_2\hat{a}_1(\Omega')\}\{r_2[\cos(\theta)\hat{a}^\dagger_{1S}(\Omega)-\sin(\theta)\hat{a}^\dagger_{2S}(\Omega)]+t_2\hat{a}^\dagger_1(\Omega)\}|0\rangle \tag{11}$$

We assume all phase and amplitude quadratures are uncorrelated, that is, all cross-quadrature terms will go to zero. This is not the case in all GW-detector situations, but it is an assumption we can make for the purposes of verifying the new FINESSE package:

$$\frac{1}{2}S(\Omega)_{PP}\delta(\Omega-\Omega')=(\hbar\omega_0 t_2 A)^2\frac{1}{2\pi}\langle 0|[(r_2\cos(\theta))^2[\hat{a}_{1S}(\Omega)\hat{a}^\dagger_{1S}(\Omega')+\hat{a}_{1S}(\Omega')\hat{a}^\dagger_{1S}(\Omega)]$$
$$+(r_2\sin(\theta))^2[\hat{a}_{2S}(\Omega)\hat{a}^\dagger_{2S}(\Omega')+\hat{a}_{2S}(\Omega')\hat{a}^\dagger_{2S}(\Omega)]+t_2^2[\hat{a}_1(\Omega)\hat{a}^\dagger_1(\Omega')+\hat{a}_1(\Omega')\hat{a}^\dagger_1(\Omega)]|0\rangle \tag{12}$$

Next we substitute in known quadrature spectral densities $S_{1S1S}=e^{-2r}$, $S_{2S2S}=e^{2r}$, and $S_{11}=1$. Here r is the squeezing factor, which is dependent on the power of the light pumped into the squeezing source.

$$\frac{1}{2}S(\Omega)_{PP}\delta(\Omega-\Omega')=(\hbar\omega_0 t_2 A)^2\frac{1}{2}\delta(\Omega-\Omega')[(r_2\cos(\theta))^2e^{-2r}+(r_2\sin(\theta))^2e^{2r}+t_2^2] \tag{13}$$

Dividing both sides of the equation by $\frac{1}{2}\delta(\Omega-\Omega')$ yields the power spectral density.

$$S(\Omega)_{PP}=(\hbar\omega_0 t_2 A)^2[(r_2\cos(\theta)e^{-r})^2+(r_2\sin(\theta)e^{r})^2+t_2^2] \tag{14}$$

4 Multi-Mode Coherent Light and Squeezed Light

4.1 Calculating Power

Coherent light in the fundamental Hermitian-Gauss mode (HG_{00}) and one higher order mode (HG_{01}) can be described in the following way immediately after leaving the laser source:

$$\hat{E}_L^{00/01}(t,z)=[A^{00}+\hat{a}_1^{00}(t)+A^{01}+\hat{a}_1^{01}(t)]\cos(\Delta(t))+[\hat{a}_2^{00}(t)+\hat{a}_2^{01}(t)]\sin(\Delta(t)) \tag{15}$$

where $\Delta(t)=\omega_0 t-kz$.

This coherent light travels a distance L to the mirror, undergoing a phase change. When it reaches the mirror it is described by:

$$\hat{E}_L^{00/01}(t',z) = [A^{00}+\hat{a}_1^{00}(t')+A^{01}+\hat{a}_1^{01}(t')]\cos(\Delta(t'))+[\hat{a}_2^{00}(t')+\hat{a}_2^{01}(t')]\sin(\Delta(t')) \tag{16}$$

Squeezed light in the same two modes, immediately after leaving the squeezing source, can be described similarly as shown below. For the purpose of comparisons with FINESSE, only the fundamental mode of the light is squeezed in the squeezing source. This will become mathematically significant later when calculating power spectral density.

$$\begin{aligned}
\hat{E}_{SL}^{00/01}(t,z) &= [\hat{a}_{1S}^{00}(t)+\hat{a}_{1S}^{01}(t)]\cos(\Delta(t)-\theta)+[\hat{a}_{2S}^{00}(t)+\hat{a}_{2S}^{01}(t)]\sin(\Delta(t)-\theta) \\
&= [\hat{a}_{1S}^{00}(t)+\hat{a}_{1S}^{01}(t)][\cos(\Delta(t))\cos(\theta)+\sin(\Delta(t))\sin(\theta)] \\
&+ [\hat{a}_{2S}^{00}(t)+\hat{a}_{2S}^{01}(t)][\sin(\Delta(t))\cos(\theta)-\cos(\Delta(t))\sin(\theta)] \\
&= [\hat{a}_{1S}^{00}(t)\cos(\theta)+\hat{a}_{1S}^{01}(t)\cos(\theta)-\hat{a}_{2S}^{00}(t)\sin(\theta)-\hat{a}_{2S}^{01}(t)\sin(\theta)]\cos(\Delta(t)) \\
&+ [\hat{a}_{2S}^{00}(t)\cos(\theta)+\hat{a}_{2S}^{01}(t)\cos(\theta)+\hat{a}_{1S}^{00}(t)\sin(\theta)+\hat{a}_{1S}^{01}(t)\sin(\theta)]\sin(\Delta(t))
\end{aligned} \tag{17}$$

where θ is the phase difference between the squeezed light and coherent light of our choosing.

We again assumed that the distance between the squeezing source and the first/bottom mirror is negligible and the mirror has a reflectivity of one. The squeezed light then travels a distance L to the second/top mirror, undergoing the same phase change as the coherent light:

$$\begin{aligned}
\hat{E}_{SL}^{00}(t',z) = {} & [\hat{a}_{1S}^{00}(t')\cos(\theta)+\hat{a}_{1S}^{01}(t')\cos(\theta)-\hat{a}_{2S}^{00}(t')\sin(\theta)-\hat{a}_{2S}^{01}(t')\sin(\theta)]\cos(\Delta(t')) \\
&+ [\hat{a}_{2S}^{00}(t')\cos(\theta)+\hat{a}_{2S}^{01}(t')\cos(\theta)+\hat{a}_{1S}^{00}(t')\sin(\theta)+\hat{a}_{1S}^{01}(t')\sin(\theta)]\sin(\Delta(t'))
\end{aligned} \tag{18}$$

Adding the coherent light transmitted through the top/second mirror (with transmissivity t_2 and reflectivity r_2) to the squeezed light reflected off of it yields the field approaching the photo detector. We assuming again that the distance between the mirror and the photo diode is negligible.

$$\begin{aligned}
\hat{E}_{LSL}^{00/01}(t',z) = {} & \{t_2[A^{00}+\hat{a}_1^{00}(t')+A^{01}+\hat{a}_1^{01}(t')] \\
&+ r_2[\hat{a}_{1S}^{00}(t')\cos(\theta)+\hat{a}_{1S}^{01}(t')\cos(\theta)-\hat{a}_{2S}^{00}(t')\sin(\theta)-\hat{a}_{2S}^{01}(t')\sin(\theta)]\}\cos(\Delta(t')) \\
&+ \{t_2[\hat{a}_2^{00}(t')+\hat{a}_2^{01}(t')]+r_2[\hat{a}_{2S}^{00}(t')\cos(\theta) \\
&+ \hat{a}_{2S}^{01}(t')\cos(\theta)+\hat{a}_{1S}^{00}(t')\sin(\theta)+\hat{a}_{1S}^{01}(t')\sin(\theta)]\}\sin(\Delta(t'))
\end{aligned} \tag{19}$$

In matrix form, the coefficients for the $\cos(\Delta(t'))$ $(\hat{b}_1^{00}(t')+\hat{b}_1^{01(t')})$ and $\sin(\Delta(t'))$ $(\hat{b}_2^{00}(t') + \hat{b}_2^{01}(t'))$ terms of the $E_{LSL}(t')$ are:

$$\begin{pmatrix} \hat{b}_1^{00}(t') \\ \hat{b}_2^{00}(t') \\ \hat{b}_1^{01}(t') \\ \hat{b}_2^{01}(t') \end{pmatrix} = r_2 \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \hat{a}_{1S}^{00}(t') \\ \hat{a}_{2S}^{00}(t') \\ \hat{a}_{1S}^{01}(t') \\ \hat{a}_{2S}^{01}(t') \end{pmatrix} + t_2 \begin{pmatrix} \hat{a}_1^{00}(t') \\ \hat{a}_2^{00}(t') \\ \hat{a}_1^{01}(t') \\ \hat{a}_2^{01}(t') \end{pmatrix} + t_2 \begin{pmatrix} A^{00} \\ 0 \\ A^{01} \\ 0 \end{pmatrix} \tag{20}$$

Now calculating power ($P = \hbar\omega_0 \hat{E}_{LSL}^2$), ignoring higher-order terms from quadratures multiplied by one another, and using double-angle formulas to remove fast-oscillating trigonometric terms:

$$\begin{aligned} P(t', z)_{00/01} =& \hbar\omega_0 t_2 \{ t_2 \frac{({A^{00}}^2 + {A^{01}}^2)}{2} + t_2[A^{00}\hat{a}_1^{00}(t') + A^{01}\hat{a}_1^{01}(t')] \\ &+ r_2[A^{00}(\hat{a}_{1S}^{00}(t')\cos(\theta) - \hat{a}_{2S}^{00}(t')\sin(\theta)) + A^{01}(\hat{a}_{1S}^{01}(t')\cos(\theta) - \hat{a}_{2S}^{01}(t')\sin(\theta))]\} \end{aligned} \tag{21}$$

Removing the constant term to focus on fluctuating power:

$$\begin{aligned} P(t', z)_{00/01} =& \hbar\omega_0 t_2 \{ t_2[A^{00}\hat{a}_1^{00}(t') + A^{01}\hat{a}_1^{01}(t')] \\ &+ r_2[A^{00}(\hat{a}_{1S}^{00}(t')\cos(\theta) - \hat{a}_{2S}^{00}(t')\sin(\theta)) + A^{01}(\hat{a}_{1S}^{01}(t')\cos(\theta) - \hat{a}_{2S}^{01}(t')\sin(\theta))]\} \end{aligned} \tag{22}$$

4.2 Calculating Power Spectral Density

First $P(t', z)$ must Fourier transformed from the time-domain into the frequency-domain.

$$\begin{aligned} P(\Omega, z)_{00/01} =& \hbar\omega_0 t_2 \{ t_2[A^{00}\hat{a}_1^{00}(\Omega) + A^{01}\hat{a}_1^{01}(\Omega)] \\ &+ r_2[A^{00}(\hat{a}_{1S}^{00}(\Omega)\cos(\theta) - \hat{a}_{2S}^{00}(\Omega)\sin(\theta)) + A^{01}(\hat{a}_{1S}^{01}(\Omega)\cos(\theta) - \hat{a}_{2S}^{01}(\Omega)\sin(\theta))]\} \end{aligned} \tag{23}$$

We substitute this equation for power into the power spectral density (PSD) equation (see Eq. 9). Again, the cross-quadrature terms are zero and several known quadrature-PSDs can be substituted into the equation. However, since only the fundamental mode of the light from the squeezing source is squeezed, higher-order squeezed light quadratures will have PSDs of one, like those of the coherent light.

$$\frac{1}{2}S(\Omega)_{PP00/01}\delta(\Omega-\Omega') =$$

$$\frac{(\hbar\omega_0 t)^2}{2}\delta(\Omega-\Omega')\{t_2^2[A^{00^2}+A^{01^2}]+r_2^2[A^{00^2}(\cos^2(\theta)e^{-2r}+\sin^2(\theta)e^{2r})+A^{01^2}(\sin^2(\theta)+\cos^2(\theta))]\}$$

$$= \frac{(\hbar\omega_0 t)^2}{2}\delta(\Omega-\Omega')\{t_2^2(A^{00^2}+A^{01^2})+r_2^2[A^{00^2}(\cos^2(\theta)e^{-2r}+\sin^2(\theta)e^{2r})+A^{01^2}]\}$$

$$= \frac{(\hbar\omega_0 t)^2}{2}\delta(\Omega-\Omega')[A^{00^2}(t_2^2+r_2^2\cos^2(\theta)e^{-2r}+r_2^2\sin^2(\theta)e^{2r})+A^{01^2}(t^2+r^2)]$$

$$= \frac{(\hbar\omega_0 t)^2}{2}\delta(\Omega-\Omega')[A^{00^2}(t_2^2+r_2^2\cos^2(\theta)e^{-2r}+r_2^2\sin^2(\theta)e^{2r})+A^{01^2}] \tag{24}$$

Dividing both sides by $\frac{1}{2}\delta(\Omega-\Omega')$ yields the spectral density for the two-mode situation:

$$S(\Omega)_{PP00/01} = (\hbar\omega_0 t)^2[A^{00^2}(t_2^2+r_2^2\cos^2(\theta)e^{-2r}+r_2^2\sin^2(\theta)e^{2r})+A^{01^2}] \tag{25}$$

For an arbitrary number of modes, not all of which need be matched between the coherent and squeezed light, the spectral density equation is:

$$S(\Omega)_{PP00/.../nm} = (\hbar\omega_0 t)^2[\sum_{matched} A^{mn^2}(t_2^2+r_2^2\cos^2(\theta)e^{-2r}+r_2^2\sin^2(\theta)e^{2r})+\sum_{mismatched} A^{mn^2}] \tag{26}$$

4.3 Comparing Analytical Results to FINESSE Simulation

To verify the multi-mode squeezing package of FINESSE, we compare the square-root of the power spectral density (known as the shot noise) from our analytical results to the same value produced by the simulation for varying phase differences (θ) between the coherent and squeezed light. The results are depicted below.

Considering the fundamental mode (HG_{00}) case:

It appears that the FINESSE simulation is well-matched with the analytical solution for all frequencies. This conclusion is further supported by the low relative error (around order 10^{-5}) between the simulated and analytical solutions:

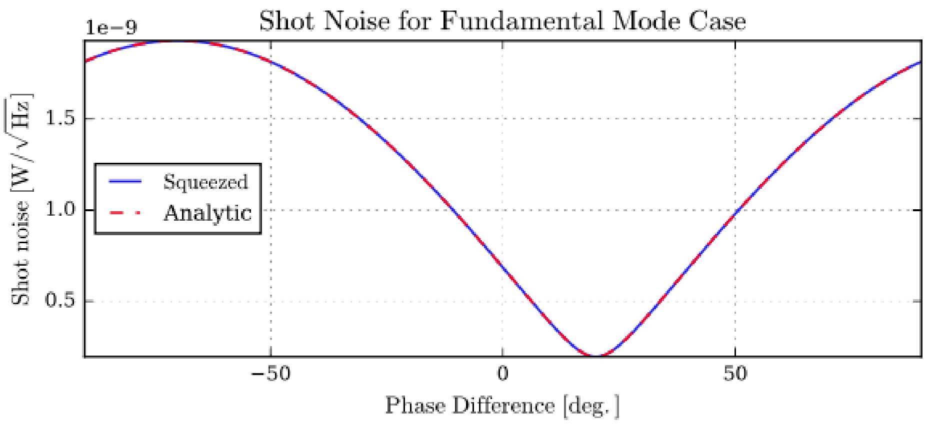

Figure 3: Analytical and FINESSE results for shot noise are plotted against phase difference for the case in which coherent light and squeezed light are both fully in the fundamental mode HG_{00}.

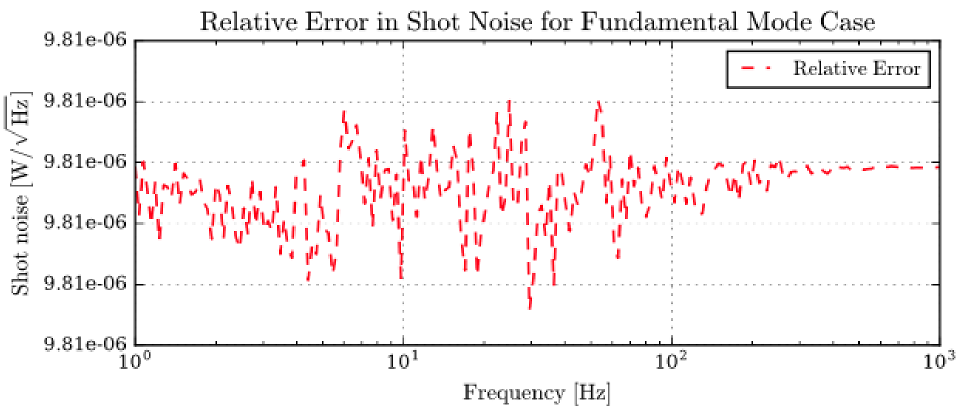

Figure 4: Relative error between analytical and FINESSE results for shot noise for fundamental mode case, plotted against phase difference.

Considering the multi-mode (HG_{00} and HG_{01}) case:

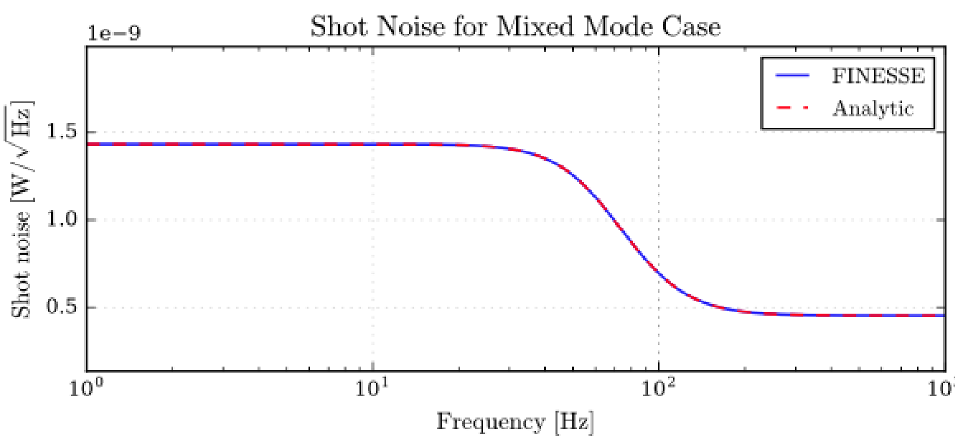

Figure 5: Analytical and FINESSE results for shot noise are plotted against phase difference for the case in which coherent light and squeezed light are both are comprised of two modes, the fundamental mode HG_{00} and one higher-order mode HG_{01}. Recall that only the fundamental mode squeezed by the non-linear crystal in the squeezing source.

It again appears that the FINESSE simulation is well-matched with the analytical solution for all frequencies. Again consider the relative error (still just under order 10^{-5}) between the two solutions:

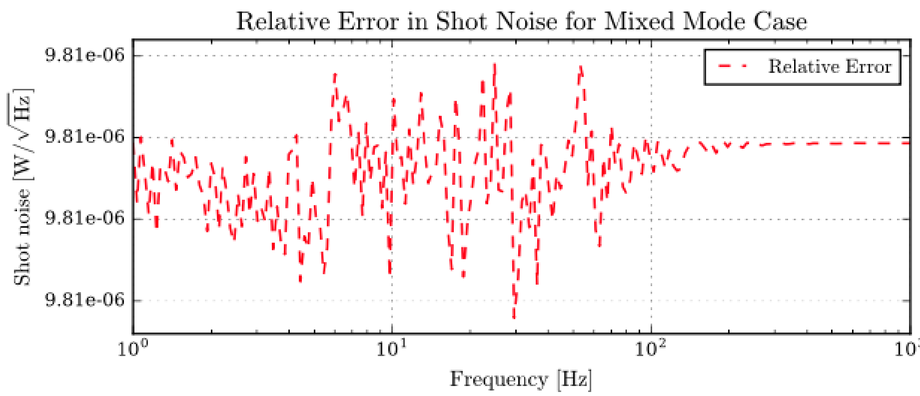

Figure 6: Relative error between analytical and FINESSE results for shot noise for multi-mode case, plotted against phase difference.

Considering the purely higher-order mode (HG_{01}) case:

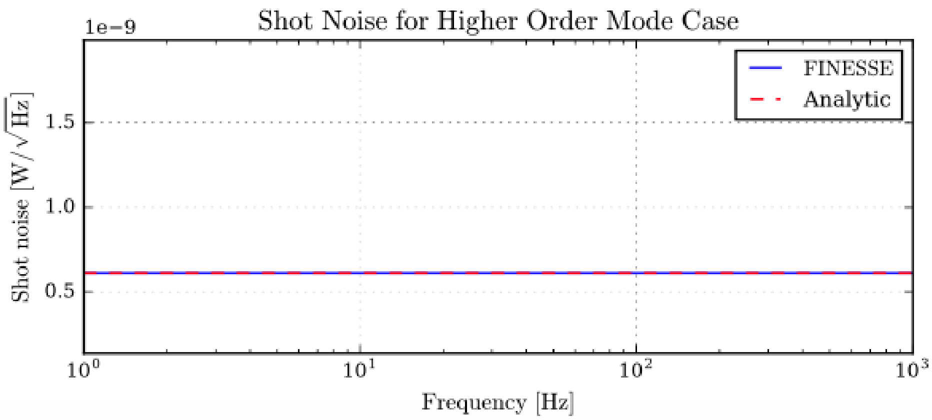

Figure 7: Analytical and FINESSE results for shot noise are plotted against phase difference for the case in which coherent light and squeezed light are both fully in the higher-order mode HG_{01}. Recall that only the fundamental mode is squeezed by the non-linear crystal in the squeezing source.

Again, the FINESSE simulation matches the analytical result well. In this case, the shot noise is constant because there is no squeezed light combining with the coherent light. This is because only the fundamental mode is squeezed in the squeezing source, but all light is in the higher-order mode in this case. Thus, this graph reflects the constant amplitude of the coherent light observed by the photo diode without the effect of squeezing. Let us again examine the relative error (nearly order 10^{-5}) between the analytic and FINESSE results:

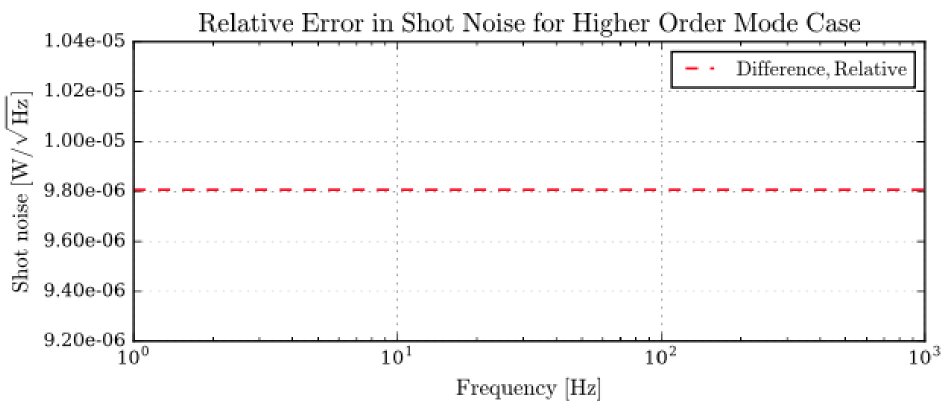

Figure 8: Relative error between analytical and FINESSE results for shot noise for higher order mode case, plotted against phase difference.

5 Single Mode Coherent Light and Squeezed Light with a Filter Cavity

5.1 Calculating Power

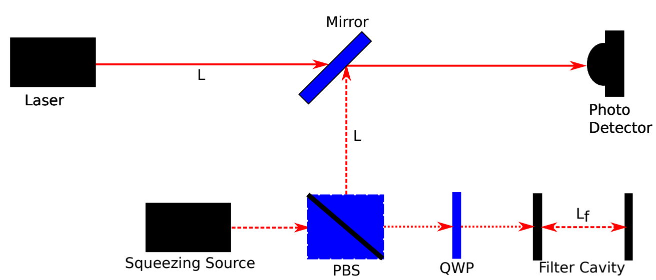

Figure 9: Squeezed light passes through a polarized beam splitter, quarter wave plate, and filter cavity before meeting light from a coherent source and entering a photo detector.

We only consider light in the fundamental Hermitian-Gauss mode (HG_{00}) in this case, so the coherent light is described by Eq. 1 immediately after exiting the laser source.

We assume that the distances between the squeezing source, polarized beam splitter (PBS), quarter wave plate (QWP), and filter cavity are negligible. We also assume that when exiting the squeezing source, the squeezed light is polarized in such a way that it is fully transmitted through the polarized beam splitter. When it goes through the QWP it experiences a $\frac{\pi}{2}$ change in either its x or y-component (perpendicular to its direction of propagation), but its z-component remains unchanged. Thus, prior to entering the filter cavity, the fundamental-mode squeezed light field is described by Eq. 2.

After exiting the filter cavity, the light passes through the QWP again, causing the same $\frac{\pi}{2}$ phase change as before. The two trips through the QWP have caused the beam to become oppositely-polarized from its original state, so it is fully reflected by the PWB upon reaching it for the second time. After traveling a distance L to the top mirror and being nearly fully reflected, it joins the coherent light that has also traveled a distance L and been transmitted (again, very little of the beam is transmitted) through the top/second mirror. At this point, the field approaching the photo diode is of the following form (we assume again that the distance between the top mirror and the photodiode is negligible):

$$\hat{E}_{LSL}(t', z) = (B + \hat{b}_1(t')) \cos(\Delta(t')) + \hat{b}_2(t') \sin(\Delta(t')) \tag{27}$$

where B is the amplitude of the field and $\hat{b}_1(t')$ and $\hat{b}_2(t')$ are its amplitude and phase quadratures, respectively.

Again, we ignore the higher-order terms that arise when quadratures are multiplied by one another when calculating the power ($P = \hbar\omega_0 \hat{E}^2_{LSL}$):

$$P(t', z) = \hbar\omega_0 [B^2 \cos^2(\Delta(t')) + 2B\hat{b}_1(t') \cos^2(\Delta(t')) + 2B\hat{b}_2(t') \cos(\Delta(t')) \sin(\Delta(t'))] \tag{28}$$

Using double angle formulas and removing the resulting fast-oscillating terms:

$$P(t', z) = \hbar\omega_0 [\frac{B^2}{2} + B\hat{b}_1(t')] \tag{29}$$

Ignoring the constant term gives the fluctuating power:

$$P(t', z) = \hbar\omega_0 B\hat{b}_1(t') \tag{30}$$

A Fourier transform yields power in terms of Ω. Note that $\hat{b}_1(\Omega)$ is composed of both coherent ($\hat{b}_C(\Omega)$) and squeezed ($\hat{b}_S(\Omega)$) light. Recognize also that the coherent laser light has been transmitted through the top mirror, so $B = t_2 A$ where t_2 is the transmissivity of that mirror.

$$\begin{aligned} P(\Omega, z) &= \hbar\omega_0 t_2 A\hat{b}_1(\Omega) \\ &= \hbar\omega_0 t_2 A[\hat{b}_{1S}(\Omega) + \hat{b}_{1C}(\Omega)] \end{aligned} \tag{31}$$

Noting that $\hat{b}_{1S}(\Omega) = \frac{\hat{b}_S(\Omega) + \hat{b}^\dagger_S(-\Omega)}{\sqrt{2}}$ and $\hat{b}_{1C}(\Omega) = \frac{\hat{b}_C(\Omega) + \hat{b}^\dagger_C(-\Omega)}{\sqrt{2}}$ by definition, this equation can be moved from the sideband-picture to the quadrature-picture:

$$P(\Omega, z) = \hbar\omega_0 t_2 A[\frac{\hat{b}_S(\Omega) + \hat{b}^\dagger_S(-\Omega)}{\sqrt{2}} + \frac{\hat{b}_C(\Omega) + \hat{b}^\dagger_C(-\Omega)}{\sqrt{2}}] \tag{32}$$

Here $\hat{b}_S(\Omega, z)$ is the annihilation operator of the squeezed light that has passed through the filter cavity and the QWP, been reflected fully by the PBS, traveled a distance L to the top mirror (with reflectivity r_2), and been reflected. Thus it can be defined as

$$\hat{b}_S(\Omega) = r_2 e^{-i\theta} \rho(\Omega) \hat{a}_S(\Omega) \tag{33}$$

and the creation operator of the squeezed light is

$$\hat{b}^\dagger_S(-\Omega) = r_2 e^{i\theta} \rho^*(-\Omega) \hat{a}^\dagger_S(-\Omega) \tag{34}$$

where $\hat{a}_S(\Omega)$ and $\hat{a}^\dagger_S(\Omega)$ are the annihilation and creation operators, respectively, of the squeezed light immediately after leaving the squeezing source. Here $\rho(\Omega)$ is the transfer function, which describes how the filter cavity affects the annihilation operator. An explicit formula for this function can be found by following the annihilation operator through the cavity.

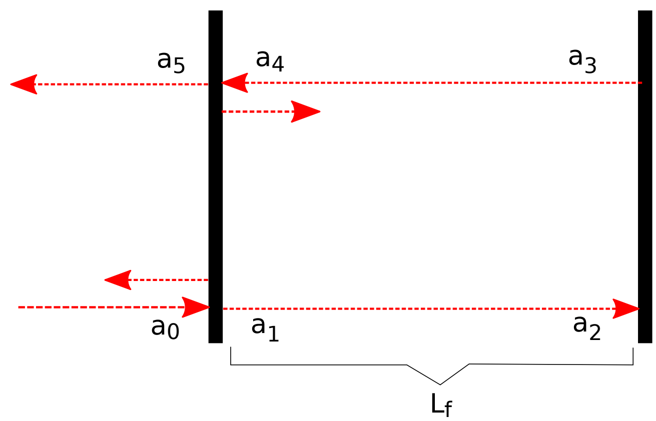

Figure 10: The annihilation operator of an electric field is shown propagating through a two-mirror filter cavity of length L_f.

Let $\hat{a}_S(t) = \hat{a}_0(t)$ be the annihilation operator of the squeezed light field just before entering the cavity, see Figure 10. Let r_1 be the reflectivity of the front mirror and t_1 its transmissivity.

Just after entering the filter cavity:

$$\hat{a}_1(t) = it_1 \hat{a}_0(t) + r_1 \hat{a}_4(t) \tag{35}$$

Approaching far mirror, the light has undergone a phase change after traveling a distance of L_f through the cavity:

$$\hat{a}_2(t) = e^{-ikL_f} \hat{a}_1(t) = e^{-ikL_f} [it_1 \hat{a}_0(t) + r_1 \hat{a}_4(t)] \tag{36}$$

Just reflected off of the far mirror, which we assume has a reflectivity of one:

$$\hat{a}_3(t) = \hat{a}_2(t) = e^{-ikL_f}[it_1\hat{a}_0(t) + r_1\hat{a}_4(t)] \tag{37}$$

Approaching the near mirror the operator has undergone another phase change from traveling the length of the cavity again:

$$\begin{aligned} &\hat{a}_4(t) = e^{-ikL_f}\hat{a}_3(t) = e^{-2ikL_f}(it_1\hat{a}_0(t) + r_1\hat{a}_4(t)) \\ &\Leftrightarrow \hat{a}_4(t) - r_1 e^{-2ikL_f}\hat{a}_4(t) = e^{-2ikL_f} it_1\hat{a}_0(t) \\ &\Leftrightarrow \hat{a}_4(t) = \frac{e^{-2ikL_f} it_1\hat{a}_0(t)}{1 - r_1 e^{-2ikL_f}} \end{aligned} \tag{38}$$

Just exited filter cavity:

$$\begin{aligned} &\hat{a}_5(t) = it_1\hat{a}_4(t) + r_1\hat{a}_0(t) = \hat{a}_0(t)(r_1 - \frac{t_1^2 e^{-2ikL_f}}{1 - r_1 e^{-2ikL_f}}) \\ &= \hat{a}_0(t)(\frac{r_1 - r_1{}^2 e^{-2ikL_f} - t_1{}^2 e^{-2ikL_f}}{1 - r_1 e^{-2ikL_f}}) \\ &= \hat{a}_0(t)(\frac{r_1 - (r_1{}^2 + t_1^2)e^{-2ikL_f}}{1 - r_1 e^{-2ikL_f}}) \\ &= \hat{a}_0(t)(\frac{r_1 - e^{-2ikL_f}}{1 - r_1 e^{-2ikL_f}}) \end{aligned} \tag{39}$$

By definition, the wave number k can be written as $\frac{\omega}{c}$ where ω is angular frequency. When the incident beam is resonant within the cavity, it should not undergo a phase change when passing through the cavity: $\hat{a}_5(t)$ should be equal to $\hat{a}_0(t)$. In that case, $e^{-2ikL_f} = 1 \Rightarrow -2ikL_f = 0 \Rightarrow k = \frac{\omega}{c} = 0 \Rightarrow \omega = 0$. The incident frequency cannot be 0Hz in any realistic setting, therefore we assign ω to be the difference between the incident frequency $\omega_0 + \Omega$ and the resonant frequency of the cavity $\omega_0 + \Omega_C$. Therefore $k = \frac{\Omega - \Omega_C}{c}$.

Thus the transfer function of the filter cavity is:

$$\rho(\Omega) = \frac{r_1 - e^{-2i(\Omega - \Omega_C)L_f/c}}{1 - r_1 e^{-2i(\Omega - \Omega_C)L_f/c}}. \tag{40}$$

The coherent light has been transmitted through the top mirror so its annihilation operator can be written as

$$\hat{b}_C(\Omega) = t_2\hat{a}_C(\Omega) \tag{41}$$

and its creation operator as

$$\hat{b}_C^\dagger(-\Omega) = t_2\hat{a}_C^\dagger(-\Omega) \tag{42}$$

where $\hat{a}_C(\Omega)$ and $\hat{a}_C^\dagger(\Omega)$ are the annihilation and creation operators, respectively, of the coherent light immediately after leaving the laser source.

Making these substitutions:

$$P(\Omega, z) = \hbar\omega_0 \frac{1}{\sqrt{2}} t_2 A[r_2 e^{-i\theta}\rho(\Omega)\hat{a}_S(\Omega) + r_2 e^{i\theta}\rho^*(-\Omega)\hat{a}_S^\dagger(-\Omega) + t_2\hat{a}_C(\Omega) + t_2\hat{a}_C^\dagger(-\Omega)]$$

$$= \hbar\omega_0 \frac{1}{\sqrt{2}} t_2 A\{r_2[e^{-i\theta}\rho(\Omega)\hat{a}_S(\Omega) + e^{i\theta}\rho^*(-\Omega)\hat{a}_S^\dagger(-\Omega)] + t_2[\hat{a}_C(\Omega) + \hat{a}_C^\dagger(-\Omega)]\}$$
$$= \hbar\omega_0 \frac{1}{\sqrt{2}} t_2 A\{r_2[e^{-i(\lambda(\Omega)+\theta)}\hat{a}_S(\Omega) + e^{i(\lambda^*(-\Omega)+\theta)}\hat{a}_S^\dagger(-\Omega)] + t_2[\hat{a}_C(\Omega) + \hat{a}_C^\dagger(-\Omega)]\} \tag{43}$$

where $\lambda(\Omega) = \arg(\rho(\Omega))$ and θ is the phase difference between the coherent and squeezed light that can be set to any desired value.

This equation can be further simplified by allowing $\Phi(\Omega) = \frac{\lambda(\Omega)-\lambda^*(-\Omega)}{2}$ and $\Gamma(\Omega) = \frac{\lambda(\Omega)+\lambda^*(-\Omega)}{2}$. This leaves:

$$P(\Omega, z) = \hbar\omega_0 \frac{1}{\sqrt{2}} t_2 A\{r_2 e^{-i\Phi(\Omega)}[e^{-i(\Gamma(\Omega)+\theta)}\hat{a}_S(\Omega) + e^{i(\Gamma(\Omega)+\theta)}\hat{a}_S^\dagger(-\Omega)] + t_2[\hat{a}_C(\Omega) + \hat{a}_C^\dagger(-\Omega)]\} \tag{44}$$

Moving the equation back into the quadrature-picture:

$$P(\Omega, z) = \hbar\omega_0 t_2 A\{r_2 e^{-i\Phi(\Omega)}[\hat{a}_{1S}(\Omega)\cos(\Gamma(\Omega)+\theta) + \hat{a}_{2S}(-\Omega)\sin(\Gamma(\Omega)+\theta)] + t_2\hat{a}_1(\Omega)\} \tag{45}$$

5.2 Calculating Power Spectral Density

Substituting this expression for power into the power spectral density equation (Eq. 9):

$$\frac{1}{2}S(\Omega)_{PP}\delta(\Omega - \Omega')$$
$$= (\hbar\omega_0 t_2 A)^2 \frac{1}{2\pi}\langle 0|\{r_2 e^{-i\Phi(\Omega)}[\hat{a}_{1S}(\Omega)\cos(\Gamma(\Omega)+\theta) + \hat{a}_{2S}(-\Omega)\sin(\Gamma(\Omega)+\theta)] + t_2\hat{a}_1(\Omega)\}$$
$$\{r_2 e^{-i\Phi^*(\Omega')}[\hat{a}_{1S}^\dagger(\Omega')\cos(\Gamma(\Omega')+\theta) + \hat{a}_{2S}^\dagger(-\Omega')\sin(\Gamma(\Omega')+\theta)] + t_2\hat{a}_1^\dagger(\Omega')\}$$
$$+ \{r_2 e^{-i\Phi(\Omega')}[\hat{a}_{1S}(\Omega')\cos(\Gamma(\Omega')+\theta) + \hat{a}_{2S}(-\Omega')\sin(\Gamma(\Omega')+\theta)] + t_2\hat{a}_1(\Omega')\}$$
$$\{r_2 e^{-i\Phi^*(\Omega)}[\hat{a}_{1S}^\dagger(\Omega)\cos(\Gamma(\Omega)+\theta) + \hat{a}_{2S}^\dagger(-\Omega)\sin(\Gamma(\Omega)+\theta)] + t_2\hat{a}_1^\dagger(\Omega)\}|0\rangle \tag{46}$$

Noting that cross-quadrature terms cancel:

$$\frac{1}{2}S(\Omega)_{PP}\delta(\Omega-\Omega') =$$
$$(\hbar\omega_0 t_2 A)^2\frac{1}{2\pi}\langle 0|\{r_2^2 e^{-i(\Phi(\Omega)-\Phi^*(\Omega'))}[\hat{a}_{1S}(\Omega)\hat{a}^\dagger_{1S}(\Omega')\cos(\Gamma(\Omega)+\theta)\cos(\Gamma(\Omega')+\theta)$$
$$+\hat{a}_{2S}(-\Omega)\hat{a}^\dagger_{2S}(-\Omega')\sin(\Gamma(\Omega)+\theta)\sin(\Gamma(\Omega')+\theta)]+t_2^2\hat{a}_1(\Omega)\hat{a}_1^\dagger(\Omega')\}$$
$$+\{r_2^2 e^{i(\Phi(\Omega)-\Phi^*(\Omega'))}[\hat{a}_{1S}(\Omega')\hat{a}^\dagger_{1S}(\Omega)\cos(\Gamma(\Omega')+\theta)\cos(\Gamma(\Omega)+\theta)$$
$$+\hat{a}_{2S}(-\Omega')\hat{a}^\dagger_{2S}(-\Omega)\sin(\Gamma(\Omega')+\theta)\sin(\Gamma(\Omega)+\theta)]+t_2^2\hat{a}_1(\Omega')\hat{a}_1^\dagger(\Omega)\}|0\rangle \tag{47}$$

It is known that $\langle 0|\hat{a}_{1S}(\Omega)\hat{a}^\dagger_{1S}(\Omega)|0\rangle = \delta(\Omega-\Omega')$.
Therefore $e^{-i(\Phi(\Omega)-\Phi^*(\Omega'))}\langle 0|\hat{a}_{1S}(\Omega)\hat{a}^\dagger_{1S}(\Omega')|0\rangle = e^{-i(\Phi(\Omega)-\Phi^*(\Omega')}\delta(\Omega-\Omega')$ is only non-zero when $\Omega=\Omega'$. Thus we assume $\Omega=\Omega'$ moving forward, which yields:

$$\frac{1}{2}S(\Omega)_{PP}\delta(\Omega-\Omega') =$$
$$(\hbar\omega_0 t_2 A)^2\frac{1}{2\pi}\langle 0|\{r_2^2[\hat{a}_{1S}(\Omega)\hat{a}^\dagger_{1S}(\Omega')\cos^2(\Gamma(\Omega)+\theta)+\hat{a}_{2S}(-\Omega)\hat{a}^\dagger_{2S}(-\Omega')\sin^2(\Gamma(\Omega)+\theta)]+t_2^2\hat{a}_1(\Omega)\hat{a}_1^\dagger(\Omega')\}$$
$$+\{r_2^2[\hat{a}_{1S}(\Omega')\hat{a}^\dagger_{1S}(\Omega)\cos^2(\Gamma(\Omega)+\theta)+\hat{a}_{2S}(-\Omega')\hat{a}^\dagger_{2S}(-\Omega)\sin^2(\Gamma(\Omega)+\theta)]+t_2^2\hat{a}_1(\Omega')\hat{a}_1^\dagger(\Omega)\}|0\rangle$$

$$=(\hbar\omega_0 t_2 A)^2\frac{1}{2\pi}\langle 0|r_2^2[\hat{a}_{1S}(\Omega)\hat{a}^\dagger_{1S}(\Omega')+\hat{a}_{1S}(\Omega')\hat{a}^\dagger_{1S}(\Omega)]\cos^2(\Gamma(\Omega)+\theta)$$
$$+[\hat{a}_{2S}(-\Omega)\hat{a}^\dagger_{2S}(-\Omega')+\hat{a}_{2S}(-\Omega')\hat{a}^\dagger_{2S}(-\Omega)]\sin^2(\Gamma(\Omega)+\theta)+t_2^2[\hat{a}_1(\Omega)\hat{a}_1^\dagger(\Omega')+\hat{a}_1(\Omega')\hat{a}_1^\dagger(\Omega)]|0\rangle \tag{48}$$

Using known values for S_{1S1S}, S_{2S2S}, and S_{11}:

$$\frac{1}{2}S(\Omega)_{PP}\delta(\Omega-\Omega') = (\hbar\omega_0 A)^2\frac{1}{2}\delta(\Omega-\Omega')\{r_2^2[e^{-2r}\cos^2(\Gamma(\Omega)+\theta)+e^{2r}\sin^2(\Gamma(\Omega)+\theta)]+t_2^2\} \tag{49}$$

Dividing both sides of the equation by $\frac{1}{2}\delta(\Omega-\Omega')$ yields the power spectral density:

$$S(\Omega)_{PP} = (\hbar\omega_0 t_2 A)^2\{r_2^2[e^{-2r}\cos^2(\Gamma(\Omega)+\theta)+e^{2r}\sin^2(\Gamma(\Omega)+\theta)]+t_2^2\} \tag{50}$$

6 Multi-Mode Coherent Light and Squeezed Light with a Filter Cavity

6.1 Calculating Power

When a Gaussian beam propagates, it picks up an extra phase factor compared to a plane wave called the Gouy phase. We were able to ignore this extra phase factor in Sections 3 and 4 because the coherent light and squeezed light traveled the same distances before meeting, so they accumulated the same Gouy phase. In Section 5, we were able to ignore the Gouy phase because even though only the squeezed light moved through the filter cavity (and thus did not travel a distance equal to the coherent light), the Gouy phase is set to be zero for the fundamental mode for our purposes. However, as this section considers a filter cavity and higher-order modes, the Gouy phase cannot be ignored, so the cavity's transfer function from Eq. 41 must be adjusted to account for the Gouy phase the squeezed light accumulates while in the filter cavity:

$$\rho(\Omega)_{mn} = \frac{r - e^{-2i[(\Omega-\Omega_C)\frac{L_f}{c} - (m+n+1)\phi(L_f)]}}{1 - re^{-2i[(\Omega-\Omega_C)\frac{L_f}{c} - (m+n+1)\phi(L_f)]}} \tag{51}$$

where $\phi(L_f)$ is the Gouy phase and L_f is the length of the filter cavity.

The expressions for the annihilation and creation operators post-filter cavity must also be updated slightly to account for multiple possible modes:

$$\hat{a}_{5_{mn}}(\Omega) = e^{i\lambda(\Omega)}\hat{a}_{0_{mn}}(\Omega) \tag{52}$$

and

$$\hat{a}_{5_{mn}}(-\Omega)^\dagger = e^{i\lambda^*(-\Omega)}\hat{a}^\dagger_{0_{mn}}(-\Omega) \tag{53}$$

where $\lambda(\Omega) = \arg\rho(\Omega)$.

When the squeezed light (after passing through the QWP and being reflected by the PBS) travels a distance L to the top mirror and is reflected it is joined by the transmitted coherent light, which has also traveled a distance L to the mirror (which has reflectivity r_2 and transmissivity t_2). Thus the annihilation and creation operators of the field approaching the photo diode (which is again assumed to be a negligible distance from the top mirror) are:

$$\hat{c}_{mn}(\Omega) = r_2 e^{i(\lambda(\Omega)+\theta)}\hat{a}_{Smn}(\Omega) + t_2\hat{a}_{mn}(\Omega) \tag{54}$$

and

$$\hat{c}^\dagger_{mn}(-\Omega) = r_2 e^{i(\lambda(-\Omega)+\theta)}\hat{a}_{Smn}(-\Omega) + t_2\hat{a}_{mn}(-\Omega) \tag{55}$$

where θ is the phase difference between the squeezed and coherent light of our choosing.

The creation and annihilation operators can be used to find the phase and amplitude quadratures of the field, shown here in matrix form:

$$\begin{pmatrix} \hat{c}_{1mn}(\Omega) \\ \hat{c}_{2mn}(\Omega) \end{pmatrix} = r_2 e^{i\Phi_{mn}(\Omega)} \begin{pmatrix} \cos(\Gamma_{mn}(\Omega)+\theta) & -\sin(\Gamma_{mn}(\Omega)+\theta) \\ \sin(\Gamma_{mn}(\Omega)+\theta) & \cos(\Gamma_{mn}(\Omega)+\theta) \end{pmatrix} \begin{pmatrix} \hat{a}_{S1mn}(\Omega) \\ \hat{a}_{S2mn}(\Omega) \end{pmatrix} + t_2 \begin{pmatrix} \hat{a}_{1mn}(\Omega) \\ \hat{a}_{2mn}(\Omega) \end{pmatrix} \tag{56}$$

For a two-mode case (HG_{00} and HG_{01}) these quadratures will be:

$$\begin{pmatrix} \hat{c}_1^{00}(\Omega) \\ \hat{c}_2^{00}(\Omega) \\ \hat{c}_1^{01}(\Omega) \\ \hat{c}_2^{01}(\Omega) \end{pmatrix} = r_2 \begin{pmatrix} e^{i\Phi^{00}(\Omega)} & e^{i\Phi^{00}(\Omega)} & e^{i\Phi^{01}(\Omega)} & e^{i\Phi^{01}(\Omega)} \end{pmatrix}$$
$$\begin{pmatrix} \cos(\Gamma^{00}(\Omega)+\theta) & -\sin(\Gamma^{00}(\Omega)+\theta) & 0 & 0 \\ \sin(\Gamma^{00}(\Omega)+\theta) & \cos(\Gamma^{00}(\Omega)+\theta) & 0 & 0 \\ 0 & 0 & \cos(\Gamma^{01}(\Omega)+\theta) & -\sin(\Gamma^{01}(\Omega)+\theta) \\ 0 & 0 & \sin(\Gamma^{01}(\Omega)+\theta) & \cos(\Gamma^{01}(\Omega)+\theta) \end{pmatrix} \begin{pmatrix} \hat{a}_{S1}^{00}(\Omega) \\ \hat{a}_{S2}^{00}(\Omega) \\ \hat{a}_{S1}^{01}(\Omega) \\ \hat{a}_{S2}^{01}(\Omega) \end{pmatrix}$$
$$+ t_2 \begin{pmatrix} \hat{a}_1^{00}(\Omega) \\ \hat{a}_2^{00}(\Omega) \\ \hat{a}_1^{01}(\Omega) \\ \hat{a}_2^{01}(\Omega) \end{pmatrix} \tag{57}$$

The field approaching the photo detector takes the form:

$$\hat{E}_{LSL}(t', z) = [C_{mn} + \hat{c}_{1mn}(t')]\cos(\Delta(t')) + \hat{c}_{2mn}(t')\sin(\Delta(t')) \tag{58}$$

where $\Delta(t') = \omega_0 t' - kz$. Here C_{mn} is the amplitude of a given mode and $\hat{c}_{1mn}(t')$ and $\hat{c}_{2mn}(t')$ are its amplitude and phase quadratures, respectively.

Solving for power:

$$P_{mn}(t', z) = \hbar\omega_0[\frac{C_{mn}^2}{2} + C_{mn}\hat{c}_{1mn}(t')] \tag{59}$$

Removing the constant term leaves the fluctuating power:

$$P_{mn}(t', z) = \hbar\omega_0 C_{mn}\hat{c}_{1mn}(t') \tag{60}$$

Fourier transforming to obtain power in terms of Ω:

$$P_{mn}(\Omega, z) = \hbar\omega_0 C_{mn}\hat{c}_{1mn}(\Omega) \tag{61}$$

In the two-mode case,

$$P_{00/01}(\Omega, z) = \hbar\omega_0[C^{00}\hat{c}_1^{00}(\Omega, z) + C^{01}\hat{c}_1^{01}(\Omega, z)] \tag{62}$$

The coherent beam has beam transmitted through the top mirror before reaching the photo-detector, so $C_{mn} = t_2 A_{mn}$ where t_2 is the transmissivity of the top mirror. We can now substitute the $c_1(\Omega)$ quadrature into the power equation.

$$\begin{aligned} P_{00/01}(\Omega, z) = \hbar\omega_0\{&t_2 A^{00}[\cos(\Gamma^{00}(\Omega) + \theta)\hat{a}_{S1}^{00}(\Omega) - \sin(\Gamma^{00}(\Omega) + \theta)\hat{a}_{S2}^{00}(\Omega)] \\ &+ t_2 A^{01}[\cos(\Gamma^{01}(\Omega) + \theta)\hat{a}_{S1}^{01}(\Omega) - \sin(\Gamma^{01}(\Omega) + \theta)\hat{a}_{S2}^{01}(\Omega)] + t_2(A^{00}\hat{a}_1^{00} + A^{01}\hat{a}_1^{01})\} \end{aligned} \tag{63}$$

6.2 Calculating Power Spectral Density

Substituting this equation for power into the power spectral density equation (Eq. 9), recognizing that cross-quadrature terms are zero, setting Ω equal to Ω' (as explained above Eq. 48 on pg. 20), and substituting in known quadrature spectral densities (note again that only the fundamental mode of the light from the squeezing source has been squeezed, so higher-order squeezed light quadratures have PSDs of one) leaves the following expression:

$$\begin{aligned} \frac{1}{2}S(\Omega)_{PP_{00/01}}\delta(\Omega - \Omega') &= \frac{(\hbar\omega_0 t_2)^2}{2}\delta(\Omega - \Omega')\{(r_2 A^{00})^2[\cos^2(\Gamma^{00}(\Omega) + \theta)e^{-2r} + \sin^2(\Gamma^{00}(\Omega) + \theta)e^{2r}] \\ &+ (r_2 A^{01})^2[\cos^2(\Gamma^{01}(\Omega) + \theta) + \sin^2(\Gamma^{01}(\Omega) + \theta)] + t_2^2[(A^{00})^2 + (A^{01})^2]\} \\ &= \frac{(\hbar\omega_0 t_2)^2}{2}\delta(\Omega - \Omega')\{{A^{00}}^2[t_2^2 + r_2^2\cos^2(\Gamma^{00}(\Omega) + \theta)e^{-2r} + r_2^2\sin^2(\Gamma^{00}(\Omega) + \theta)e^{2r}] + {A^{01}}^2[t_2^2 + r_2^2]\} \\ &= \frac{(\hbar\omega_0 t_2)^2}{2}\delta(\Omega - \Omega')\{{A^{00}}^2[t_2^2 + r_2^2\cos^2(\Gamma^{00}(\Omega) + \theta)e^{-2r} + r_2^2\sin^2(\Gamma^{00}(\Omega) + \theta)e^{2r}] + {A^{01}}^2\} \end{aligned} \tag{64}$$

Dividing both sides of the equation by $\frac{1}{2}\delta(\Omega - \Omega')$ yields the power spectral density:

$$S(\Omega)_{PP_{00/01}} = (\hbar\omega_0 t_2)^2[{A^{00}}^2(t_2^2 + r_2^2\cos^2(\Gamma^{00}(\Omega) + \theta)e^{-2r} + r_2^2\sin^2(\Gamma^{00}(\Omega) + \theta)e^{2r}) + {A^{01}}^2] \tag{65}$$

For an arbitrary number of modes, not all of which need be matched between the coherent and squeezed light, the spectral density equation is:

$$\begin{aligned} &S(\Omega)_{PP_{00/../nm}} = \\ &(\hbar\omega_0 t_2)^2[\sum_{matched} A^{nm2}(t_2^2 + r_2^2\cos^2(\Gamma^{nm}(\Omega) + \theta)e^{-2r} + r_2^2\sin^2(\Gamma^{nm}(\Omega) + \theta)e^{2r}) + \sum_{mismatched} A^{nm2}] \end{aligned} \tag{66}$$

6.3 Comparing Analytical Results to FINESSE Simulation

In order to verify the multi-mode squeezing package of FINESSE for the cases including a filter cavity, we compare the shot noise from our analytical results to the same value produced by the simulation for varying sideband frequencies of incident squeezed light. We considering various frequencies instead of various phase differences (as we did in the case without a filter cavity) because the addition of a filter cavity causes squeezing to become dependent on the sideband frequency of the light from the squeezing source. The results are depicted below.

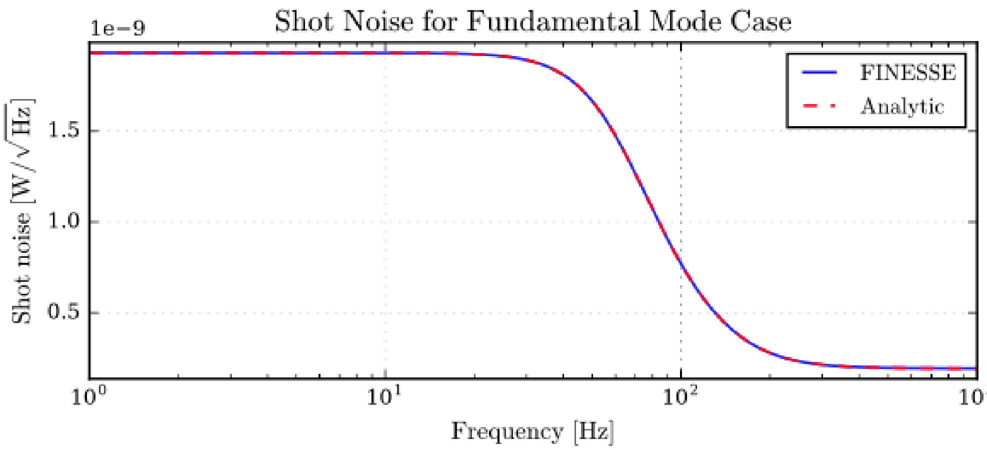

Figure 11: Analytical and FINESSE results for shot noise are plotted against frequency for the case in which coherent light and squeezed light are both fully in the fundamental mode HG_{00}.

It appears that the FINESSE simulation is well-matched with the analytical solution for all frequencies. This conclusion is further supported by the low (order 10^{-5}) relative error between the simulated and analytical solutions:

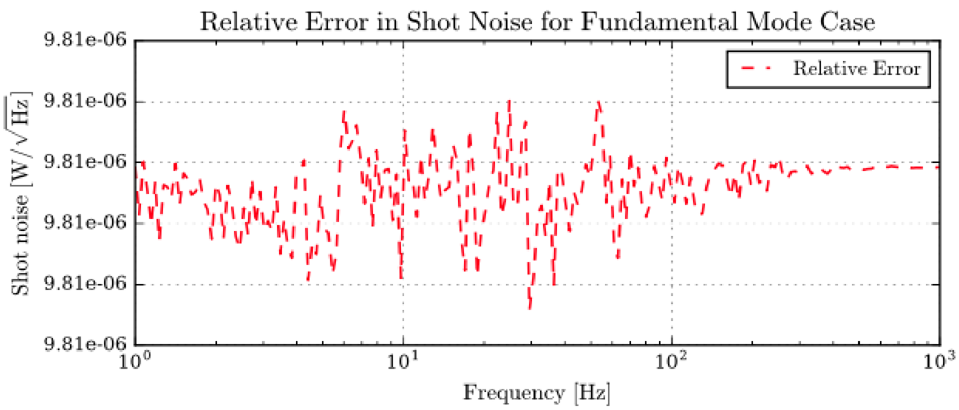

Figure 12: Relative error between analytical and FINESSE results for shot noise for fundamental mode case, plotted against frequency.

Considering the mixed-mode (HG_{00} and HG_{01}) case:

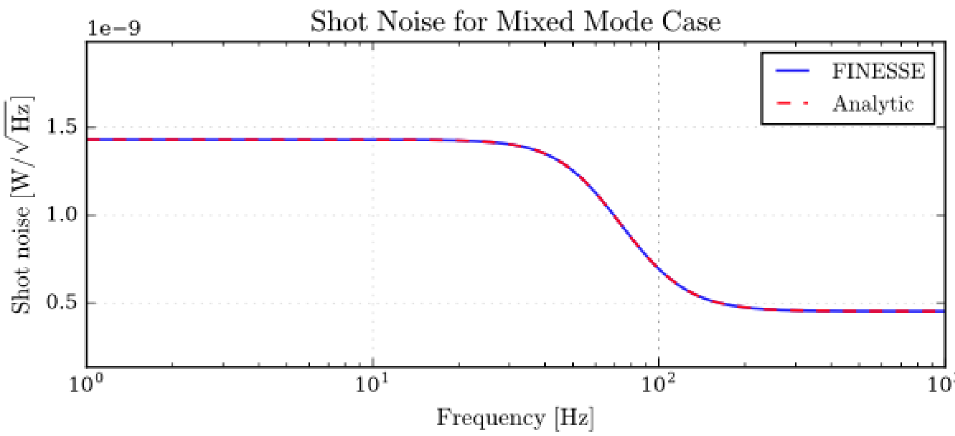

Figure 13: Analytical and FINESSE results for shot noise are plotted against frequency for the case in which coherent light and squeezed light are both are comprised of two modes, the fundamental mode HG_{00} and one higher-order mode HG_{01}. Recall that only the fundamental mode of light from the squeezing source is squeezed by the non-linear crystal.

It again appears that the FINESSE simulation is well-matched with the analytical solution for all frequencies. Again considering the relative error (order 10^{-5}) between the two solutions:

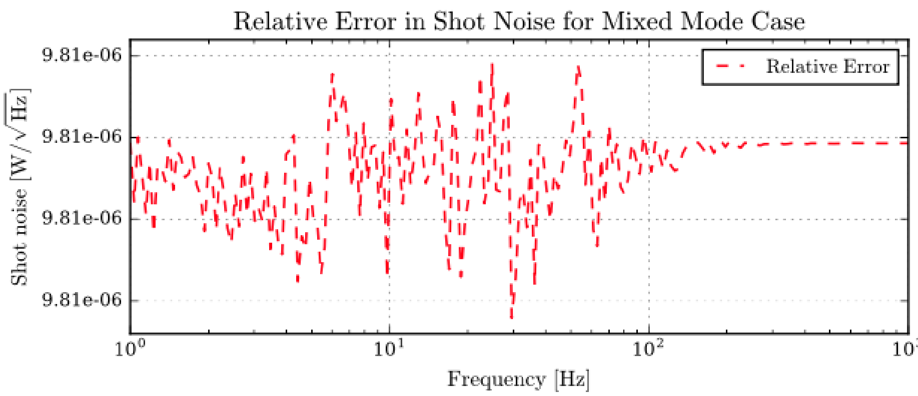

Figure 14: Relative error between analytical and FINESSE results for shot noise for the mixed-mode case, plotted against frequency.

Considering the higher-order mode (HG_{01}) case:

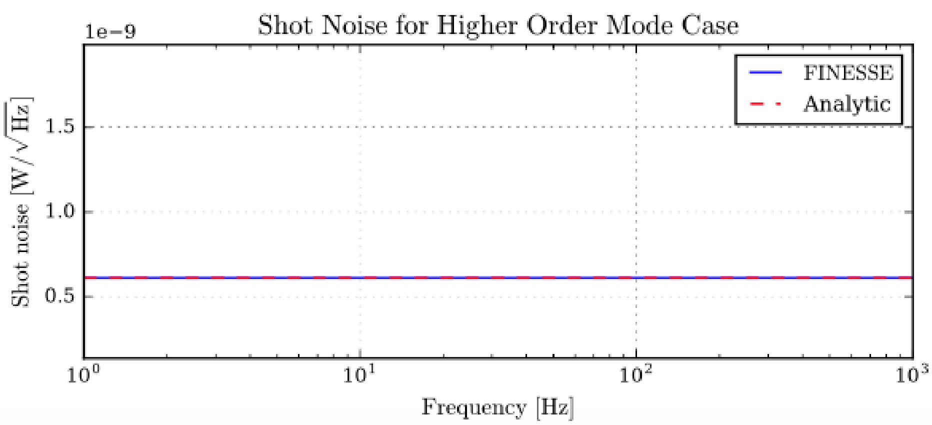

Figure 15: Analytical and FINESSE results for shot noise are plotted against phase difference for the case in which coherent light and squeezed light are both fully in the higher-order mode HG_{01}. Recall that only the fundamental mode of light from the squeezing source is squeezed by the non-linear crystal.

Again, the FINESSE simulation matches the analytical result well. In this case, the shot noise is constant because there is no squeezing light combining with the coherent light. This is because only the fundamental mode of the light from the squeezing source is squeezed, but all of the light is in the higher-order mode in this case. Thus, this graph reflects the constant amplitude of the coherent light observed by the photo diode without the effect of squeezing. Let us again examine the relative error (order 10^{-5}) between the analytic and FINESSE results:

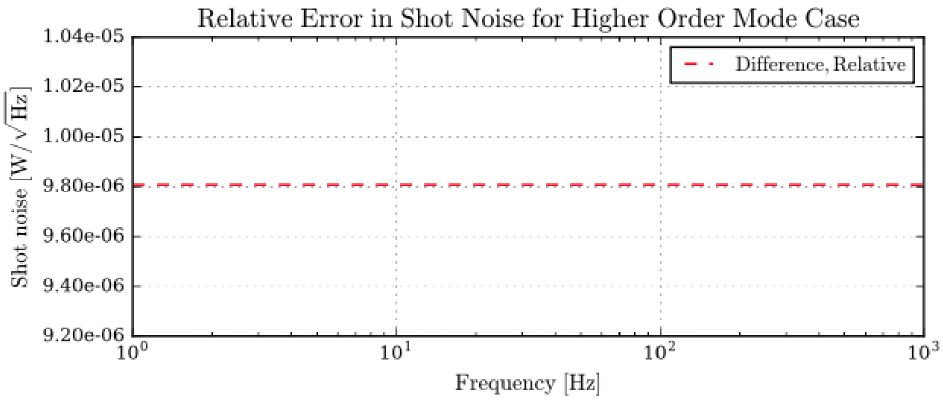

Figure 16: Relative error between analytical and FINESSE results for shot noise for higher-order mode case, plotted against phase difference.

7 Conclusion

While our results were generally compatible with those of the FINESSE simulation, we did notice small differences of order 10^{-5}, especially in the lower frequency range of the situations in Section 6. While searching for the source of this difference, we managed to uncover an error in the source code of FINESSE. Planck's constant, $\hbar$, was defined slightly inaccurately as $\hbar = 6.6262e - 34$ instead of the more correct value $\hbar = 6.62607004e - 34$. This error has likely been in the source code since the program's inception nearly 15 years ago and simply had not become evident until these trials. After correcting this error, the difference between our analytical results and those of the FINESSE simulation shrank to order 10^{-11} (the shape of the relative error graphs remained unchanged). We believe this remaining difference is likely a result of small numerical errors due to the software's need to truncate values for storage purposes. Regardless, we consider the results of FINESSE to be close enough to the analytical results to declare confidence in the accuracy of the multi-mode squeezing package. The newest package of FINESSE can now be released to the public and used to run more advanced simulations in the field. As FINESSE has already been of use in many past advancements in gravitational physics, many of which have allowed for improvements to LIGO, this new package should be of use in future discoveries in the field.

8 Acknowledgements and References

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My references are as follows:

Chelkowski, Simon. Squeezed Light and Laser Interferometric Gravitational Wave Detectors. Thesis. Hannover, University, Diss, 1976. Wilhelm Leibniz Universita t Hannover: Hannover, n.d. Print.

Freise, Andreas, Kenneth A. Strain, Daniel Brown, and Charlotte Bond. "Interferometer Techniques for Gravitational-Wave Detection." Living Reviews in Relativity Living Rev. Relativity 13 (2010): n. pag. Web.

"Random Processes." Applications of Classical Physics. Caltech: Pasadena, 2012. 1-56. Print.

Y. Chen and H. Miao. Quantum Theory of Laser-Interferometer GW Detectors. Advanced Gravitational Wave Detectors Ed. D. G. Blair, L. Ju, C. Zhao, and E. J. Howell Cambridge University Press: Cambridge, 2012. 1-23. Print.