Quantized Cosmological Constant in One-Dimensional Matter Coupled Quantum Gravity

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Abstract

Coupling an interacting quantum mechanical system to gravity in one dimension requires the cosmological constant to belong to the matter energy spectrum and thus to be quantized, even though one-dimensional gravity is free of any quantum dynamics. This general fact is illustrated through some simple examples. The physical projector quantization approach readily leads to the correct representation of such systems, whereas other methods relying on gauge fixing methods are often plagued by Gribov problems. Whether such a quantization of the cosmological constant extends to higher dimensional matter-gravity coupled quantum systems is clearly a fascinating open issue.

Dedicated to John R. Klauder
on the occasion of his 70$^{\text{th}}$ *birthday*

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1 Introduction

Though I must be a latecomer amongst John's circle of friends, it is without difficulty that one comes to recognize John's generous open-hearted enthusiasm for the ordinary as much as for the hidden aspects of the human and scientific adventure. If one had to characterize in just a few words John's secret, I believe that "always staying young at heart" comes quite very close to it. It thus gives me great pleasure to be associated with this tribute to John. My regret is not to be able to attend the organized celebration, but at least I hope that John will appreciate this little "present" in the form of a small contribution on a topic which focuses a good deal of his interests in recent years, and to which he has made a lasting contribution in the form of the physical projector for the quantization of constrained dynamics[1].

As a matter of fact, this note grew out from a remark John once made to me, in reference to a referee's criticism calling into question the relevance of the physical projector. As often with John's simple illustrations of a point at hand, they go deep into the heart of an issue, and in that instance it was to display the often overlooked subtleties surrounding the problem of gauge fixing in systems with local symmetries, including Faddeev's reduced phase space. In fact, this is the very issue which also led to our first meeting in 1996. I thus propose simply to offer in return to John the little system that he lent me for some time, but fully dressed up, hopefully leading to a physical insight concerning a burning issue in contemporary physics, namely that of the cosmological constant in quantum gravity[2]. Indeed, John's example turns out to belong to a general class of quantum mechanical models in which one-dimensional quantum gravity is coupled to interacting quantum matter. What is rather intriguing in such a circumstance is that the quantization of such systems is physically consistent provided only the cosmological constant takes a specific value within the energy spectrum of the quantized matter sector, in spite of the lack of any dynamics, whether classical or quantum, in the gravity sector. Clearly, whether such a feature survives for quantum gravity in higher dimensions when coupled to interacting quantum matter is an open issue of great interest, and John may possibly be enticed into applying his physical insight to it as well, given his recent strong interest in quantum gravity[3].

Even though this conclusion as to a quantized cosmological constant in one-dimensional quantum gravity is very general and applies whatever the nature of the interacting quantum matter sector, in the present note we shall only consider a matter sector described by cartesian degrees of freedom. Nonetheless, as the reader will readily realize, the analysis to be presented easily extends to more general cases, such as for instance quantum mechanical nonlinear sigma models whose degrees of freedom take values in some curved and/or compact manifold. As a matter of fact, the sole mention in the literature known to this author of a quantized cosmological constant in the manner discussed here[1] is that of Ref.[4], in which the interacting quantum mechanical matter sector is the nonrelativistic hydrogen atom, again a specific example of the general construction to be discussed presently.

The note is organized as follows. In Sect.2, general quantum mechanical systems, as yet uncoupled to gravity, are considered. In Sect.3, these systems are coupled to one-dimensional gravity and then quantized following Dirac's general approach[6], in order to identify their physical sector through John's physical projector[1]. It is here that the requirement of a quantized cosmological constant arises explicitly. Sect.4 then addresses the same issues, this time from Faddeev's reduced phase space approach, to conclude that this method of fixing the reparametrization gauge freedom is ill-fated in these systems. In Sect.5, the Hamiltonian BSRT methods are also brought to bear on the same systems, to conclude once again[7, 8] that even though these methods lead, by construction, to gauge invariant quantities, it is only for an admissible gauge fixing procedure that the correct physical content is properly identified. Sect.6 briefly presents some concluding remarks.

2 The Uncoupled Matter System

Quite generally, let us consider an arbitrary classical system whose Hamiltonian formulation relies on canonical phase space degrees of freedom $(q^n(t), p_n(t))$ $(n = 1, 2, \cdots, N)$, with the canonical bracket structure $\{q^n(t), p_m(t)\} = \delta_{n,m}$, whose time evolution is generated by some Hamiltonian function $H(q^n, p_n)$. The dynamics of such a system also follows from the variational principle applied to the first-order phase space Hamiltonian action

$$S\left[q^n, p_n\right] = \int dt\ \left[\dot{q}^n p_n - H(q^n, p_n)\right]\ . \tag{1}$$

As indicated previously, we shall assume that the coordinates $q^n(t)$ define a cartesian parametrization of the configuration space of the system, with $q^n(t)$ and $p_n(t)$ thus taking all possible real values, even though the construction and its conclusions to be presented in Sect.3 remain valid whatever the choice of matter sector.

In the following, when considering examples, we shall restrict to two specific cases; on the one hand, the single one-dimensional harmonic oscillator of mass m and angular frequency ω, and on the other hand, the N-dimensional euclidean spherical harmonic oscillator with the same parameters.

[1] The possibility of a quantized cosmological constant is also mentioned for instance in Ref.[5] based on other arguments.

Within this general framework, the classical equations of motion are

$$\dot{q}^n = \frac{\partial H}{\partial p_n} \quad , \quad \dot{p}_n = -\frac{\partial H}{\partial q^n} \ . \tag{2}$$

For instance, given initial values $q^n_i = q^n(t_i)$ and $p_{n,i} = p_n(t_i)$, the solution to these equations is of the form

$$q^n(t) = Q^n(t; q^n_i, p_{n,i}) \quad , \quad p_n(t) = P_n(t; q^n_i, p_{n,i}) \ , \tag{3}$$

while the "energy"[2] of the system is given by the conserved value of the Hamiltonian,

$$E(q^n_i, p_{n,i}) = H(q^n_i, p_{n,i}) \ . \tag{4}$$

Here, $Q^n(t; q^n_i, p_{n,i})$ and $P_n(t; q^n_i, p_{n,i})$ are specific functions of time, also dependent on the initial values for the phase space degrees of freedom.

As is well known, the Hamiltonian equation of motion for $\dot{q}^n$ may be used to reduce the conjugate momenta $p_n = p_n(q^n, \dot{q}^n)$ and obtain the Lagrangian variational principle for the same dynamics,

$$S[q^n] = \int dt\, L(q^n, \dot{q}^n) \quad , \quad L\left(q^n, \dot{q}^n\right) = \dot{q}^n p_n(q^n, \dot{q}^n) - H\left(q^n, p_n(q^n, \dot{q}^n)\right) \ , \tag{5}$$

as well as the Euler-Lagrange equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^n} - \frac{\partial L}{\partial q^n} = 0 \ . \tag{6}$$

This reduction is especially simple when the conjugate momenta dynamics separates through an Hamiltonian of the form $H = \sum_n p_n^2/(2m_n) + V(q^n)$. Examples considered further on belong to this latter characterization.

Within this Lagrangian formulation, a choice of boundary values which is often more convenient than the one above is such that the configuration space values $q^n(t)$ are specified both at some initial time $t = t_i$ as well as at some final time $t = t_f$, $q^n_{i,f} = q^n(t_{i,f})$. Nonetheless, knowledge of the solutions $Q^n(t; q^n_i, p_{n,i})$ and $P_n(t; q^n_i, p_{n,i})$ enables one in principle to construct the solutions to the Euler-Lagrange equations of motion for this alternative choice of boundary conditions.

The path towards quantization to be taken in this note is that of the abstract canonical quantization of the above Hamiltonian formulation. Hence, the quantum system is represented through a linear representation space $|\psi>$ of the abstract Heisenberg algebra $[\hat{q}^n(t_0), \hat{p}_m(t_0)] = i\hbar\delta_{n,m}$, equiped with an hermitean inner product $< .|. >$ such that all these operators $\hat{q}^n(t_0)$ and $\hat{p}_n(t_0)$ be—in the best of cases—self-adjoint[3]. Finally, time evolution is generated by Schrödinger's equation

$$i\hbar\frac{d}{dt}|\psi, t> = \hat{H}|\psi, t> \tag{7}$$

where $\hat{H} = \hat{H}(\hat{q}^n(t_0), \hat{p}_n(t_0))$ is a quantum Hamiltonian operator in correspondence with the classical one, $H(q^n, p_n)$, and defined in such a manner that it be self-adjoint as well.

[2]Indeed, this quantity measures the actual energy of the system only when the time evolution parameter t coincides with the physical time.

[3]Canonical quantization is performed in the Schrödinger picture, so that t_0 here stands for a reference time at which quantization is performed for the operator degrees of freedom.

For physical consistency, we shall assume that the $\hat{H}$-spectrum is bounded below. Otherwise, just for the sake of definiteness in our discussion, we shall further assume that this spectrum be discrete[4],

$$\hat{H}|E_n,\alpha_n>=E_n|E_n,\alpha_n> \quad , \quad \alpha_n=1,2,\cdots,d_n \quad , \quad n=0,1,2,\cdots \ , \tag{8}$$

with a degeneracy $d_n \ge 1$ at each of the energy levels E_n $(n=0,1,2,\cdots)$, and with a choice of orthonormalized energy eigenstates

$$<E_n,\alpha_n|E_m,\alpha_m>=\delta_{n,m}\,\delta_{\alpha_n,\alpha_m} \ , \tag{9}$$

which implies

$$\mathbb{1}=\sum_{n,\alpha_n}|E_n,\alpha_n><E_n,\alpha_n| \ . \tag{10}$$

Consequently, the Hamiltonian operator possesses the spectral representation

$$\hat{H}=\sum_{n,\alpha_n}|E_n,\alpha_n>E_n<E_n,\alpha_n| \ , \tag{11}$$

while the time evolution operator $\hat{U}(t_2,t_1)=e^{-i/\hbar(t_2-t_1)\hat{H}}$ of the quantum system simply reads

$$\hat{U}(t_2,t_1)=\sum_{n,\alpha_n}|E_n,\alpha_n>\ e^{-i/\hbar(t_2-t_1)E_n}\ <E_n,\alpha_n| \ . \tag{12}$$

Hence, given any initial state

$$|\psi,t=t_i>=\sum_{n,\alpha_n}|E_n,\alpha_n>\psi_{n,\alpha_n} \quad , \quad \psi_{n,\alpha_n}=<E_n,\alpha_n|\psi,t=t_i> \ , \tag{13}$$

the general solution to Schrödinger's equation is

$$|\psi,t>=\hat{U}(t,t_i)|\psi,t_i>=\sum_{n,\alpha_n}|E_n,\alpha_n>e^{-i/\hbar E_n(t-t_i)}\psi_{n,\alpha_n} \ . \tag{14}$$

It should be clear how any (of the well known) quantum mechanical system(s) may be brought into correspondence with such a general description, beginning with the ordinary harmonic oscillator in its Fock space representation.

3 The Gravity Coupled System

Given any general mechanical system as described in Sect.2, let us now consider the following first-order action principle,

$$S[q^n,p_n;\lambda]=\int dt\ \left[\dot{q}^n p_n-\lambda\left(H(q^n,p_n)-\Lambda\right)\right] \ , \tag{15}$$

where $\lambda(t)$ is an arbitrary function of t and Λ an arbitrary real constant parameter. Clearly, $\lambda(t)$ is a Lagrange multiplier for a constraint on phase space, namely $\phi(q^n,p_n)=0$, with

$$\phi(q^n,p_n)=H(q^n,p_n)-\Lambda \ , \tag{16}$$

[4]The quantization of the cosmological constant applies whether the matter energy spectrum is discrete or continuous, or a mixture of both.

while the Hamiltonian equations of motion now read

$$\dot{q}^n = \lambda\frac{\partial H}{\partial p_n} \quad , \quad \dot{p}_n = -\lambda\frac{\partial H}{\partial q^n} \ , \tag{17}$$

whose solutions are subjected to the constraint $H(q^n, p_n) = \Lambda$. As a matter of fact, (15) defines a constrained system[6, 8] whose phase space coordinates (q^n, p_n) are canonical, $\{q^n, p_m\} = \delta_{n,m}$, whose time evolution is generated by the total Hamiltonian $H_T = \lambda(H - \Lambda)$, and which possesses the gauge invariance degree of freedom under arbitrary local reparametrizations in t generated by the first-class constraint $\phi = H - \Lambda$. The associated first-class Hamiltonian in fact vanishes identically, as befits a reparametrization invariant system, so that the total Hamiltonian which generates time dependency is indeed "pure gauge", $H_T = \lambda\phi$, while the constraint $\phi = 0$ that classical solutions must meet expresses their gauge invariance under local time reparametrizations. The fact that (15) does define the coupling of the matter system described by $H(q^n, p_n)$ to gravity in a one-dimensional "spacetime" with Λ being a cosmological constant may be justified from complementary viewpoints. In particular, $\lambda(t)$ is nothing but a world-line einbein, with $\lambda^2(t)$ defining an intrinsic world-line metric[5].

First, given initial values q_i^n and $p_{n,i}$ at $t = t_i$ for the phase space degrees of freedom, let us construct the general solutions to the equations of motion (17) whatever the choice for the Lagrange multiplier function $\lambda(t)$. For this purpose, consider the function $\tau(t)$, with $\tau_i = t_i$, such that

$$\tau - \tau_i = \int_{t_i}^{t} dt'\,\lambda(t') \quad , \quad \tau(t_i) = t_i = \tau_i \ . \tag{18}$$

It then follows that in terms of this variable τ, the equations of motion (17) reduce to those of the uncoupled system,

$$\frac{dq^n}{d\tau} = \frac{\partial H}{\partial p_n} \quad , \quad \frac{dp_n}{d\tau} = -\frac{\partial H}{\partial q^n} \ , \tag{19}$$

while the first-order action (15) reads

$$S[q^n, p_n; \lambda] = \int d\tau \left[\frac{dq^n}{d\tau}p_n - (H(q^n, p_n) - \Lambda)\right] \ . \tag{20}$$

Consequently, irrespective of the choice for $\lambda(t)$, the general solution to the equations of motion associated to the initial values $(q_i^n, p_{n,i})$ is of the same form as that for the uncoupled system,

$$q^n(t) = Q^n(\tau(t); q_i^n, p_{n,i}) \quad , \quad p^n(t) = P^n(\tau(t); q_i^n, p_{n,i}) \ , \tag{21}$$

with however the time dependency on t substituted by that on the parameter $\tau(t)$, and the further restriction that the "energy" E of the solution must coincide with the cosmological constant value Λ,

$$E(q_i^n, p_{n,i}) = H(q_i^n, p_{n,i}) = \Lambda \ . \tag{22}$$

In other words, irrespective of the choice of world-line (re)parametrization characterized by the function $\tau(t)$ such that $\tau(t_i) = t_i$, and thus associated to the Lagrange multiplier $\lambda(t) = d\tau(t)/dt$, the solution is given by the same functions $Q^n(\tau; q_i^n, p_{n,i})$ and $P_n(\tau; q_i^n, p_{n,i})$ as those of the uncoupled system, with however the restriction that the cosmological constant Λ must belong to the H-spectrum and that the

[5]Note that the classes of models considered here include the scalar relativistic particle propagating in a spacetime of whatever geometry, the parameter Λ being directly related to the particle's squared mass.

energy E of the solution must coincide with Λ. The dependency of the general solution to the equations of motion on an arbitrary function $\tau(t)$ of time is the manifest hallmark of a system which is gauge invariant under local reparametrizations in one dimension, namely in the time parameter. The physical content of such systems is thus that which is totally independent of the world-line parametrization, as are for example the relations that exist among the phase space degrees of freedom q^n and p_n independently of t but dependent on the initial values q^n_i and $p_{n,i}$, such as the energy $E(q^n_i, p_{n,i}) = \Lambda$.

The fact that we are indeed dealing with a theory of one-dimensional gravity coupled to matter dynamics may also be justified from another viewpoint. Applying the Hamiltonian reduction of the conjugate momenta p_n discussed in Sect.2 but based this time on the coupled action (15) and the associated equations of motion (17), one readily concludes that the Lagrangian formulation of the coupled system is,

$$S[q^n;\lambda] = \int d\tau\,\left[L\left(q^n, \frac{dq^n}{d\tau}\right) + \Lambda\right] = \int dt\,\lambda(t)\,\left[L\left(q^n, \frac{1}{\lambda(t)}\frac{dq^n}{dt}\right) + \Lambda\right]\ . \tag{23}$$

In this form, it is clear that we are indeed dealing with an intrinsic world-line metric defined by the invariant line element

$$ds^2 = dt^2\lambda^2(t)\ , \tag{24}$$

with $\lambda(t)$ and $\lambda^2(t)$ as einbein and metric structures, and which are coupled in a reparametrization invariant way to the matter Lagrangian as well as to a world-line cosmological term Λ.

Finally, let us consider the local gauge invariance properties in the Hamiltonian formulation. Since $\phi = H - \Lambda$ is the generator for the associated infinitesimal transformations in phase space, their explicit expression reads

$$\delta_\epsilon q^n = \epsilon\,\{q^n,\phi\} = \epsilon\,\frac{\partial H}{\partial p_n}\quad,\quad \delta_\epsilon p_n = -\epsilon\,\{p^n,\phi\} = \epsilon\,\frac{\partial H}{\partial q^n}\quad,\quad \delta_\epsilon\lambda = \dot\epsilon\ , \tag{25}$$

where $\epsilon(t)$ is some arbitrary function. Indeed, the variation of the first-order action (15) is then a surface term,

$$\delta_\epsilon S[q^n,p_n;\lambda] = \int dt\,\frac{d}{dt}\left[\epsilon\left(p_n\frac{\partial H}{\partial p_n} - H + \Lambda\right)\right]\ , \tag{26}$$

thus expressing the fact that the transformations (25) do define a symmetry of the system, namely world-line reparametrization invariance of a general matter coupled one-dimensional gravity system.

The abstract canonical quantization of the system is straightforward enough. The gravitational sector being dynamics free, the space of quantum states is that of the matter sector, as described in Sect.2, whereas no further quantum operators are associated to the gravitational sector. In particular, the arbitrary Lagrange multiplier function $\lambda(t)$ is not related to a quantum operator, and still serves the sole purpose of parametrizing the gauge freedom related to the choice of world-line parametrization. Compared to the uncoupled quantum system, the only modification is that time dependency of states is generated through a Schrödinger equation which now reads

$$i\hbar\frac{d}{dt}|\psi,t>=\lambda(t)\left[\hat H - \Lambda\right]\,|\psi,t>\ , \tag{27}$$

or equivalently, in terms of the function $\tau(t)$,

$$i\hbar\frac{d}{d\tau}|\psi,t(\tau)>=\left[\hat H - \Lambda\right]\,|\psi,t(\tau)>\ , \tag{28}$$

thus making manifest yet again the fact that $\lambda(t)$ indeed parametrizes the freedom in world-line parametrizations.

Given an arbitrary initial state $|\psi, t_i> = \sum_{n,\alpha_n} |E_n, \alpha_n > \psi_{n,\alpha_n}$, the solution to the Schrödinger equation is thus now

$$|\psi, t> = \sum_{n,\alpha_n} |E_n, \alpha_n > \, e^{-i/\hbar (E_n - \Lambda) \int_{t_i}^{t} dt' \lambda(t')} \psi_{n,\alpha_n} \ . \tag{29}$$

More generally, the quantum time evolution operator of the quantized system is

$$\hat{U}(t_2, t_1) = \sum_{n,\alpha_n} |E_n, \alpha_n > \, e^{-i/\hbar (E_n - \Lambda) \ \int_{t_1}^{t_2} dt \lambda(t)} < E_n, \alpha_n| \ . \tag{30}$$

However, all quantum states of the system may not be regarded as being physical, *i.e.* gauge invariants states, but only those[6] that are annihilated by the gauge generator $\hat{\phi} = \hat{H} - \Lambda$,

$$\left[\hat{H} - \Lambda\right] |\psi_{\rm phys}> = 0 \ . \tag{31}$$

When combined with the Schrödinger equation, note that this restriction implies that physical states must be independent of time, namely they are indeed independent of the world-line parametrization as it should.

On the other hand, we also recover for quantum physical states a situation identical to that for the classical gauge invariant solutions, namely the fact that in order for the physical content of the system not to be void, the cosmological constant Λ must belong to the $\hat{H}$-spectrum of the matter sector,

$$\Lambda = E_{n_0} \ , \tag{32}$$

for some specific integer value of n_0. In other words, one-dimensional quantum gravity coupled to interacting quantum matter is physical provided only the cosmological constant is quantized with a value belonging to the energy spectrum of the quantum matter sector. Clearly, this result is very general, whatever the quantum matter sector. If the latter possesses only a discrete spectrum, Λ is quantized within that discrete spectrum, and likewise for a continuous domain in the $\hat{H}$-spectrum.

Let us hence assume that the value Λ coincides with one of the energy eigenvalues E_{n_0}. Consequently, the subspace of physical quantum states is spanned by the the states $|E_{n_0}, \alpha_{n_0} >$ with a degeneracy d_{n_0},

$$|\psi_{\rm phys}> = \sum_{\alpha_{n_0}} |E_{n_0}, \alpha_{n_0} > \ \psi_{\alpha_{n_0}} \quad , \quad \psi_{\alpha_{n_0}} = < E_{n_0}, \alpha_{n_0} | \psi_{\rm phys} > \ , \tag{33}$$

to which the following physical projector[1] is thus associated,

$$\boldsymbol{E} = \sum_{\alpha_{n_0}} |E_{n_0}, \alpha_{n_0} >< E_{n_0}, \alpha_{n_0}| \quad , \quad \boldsymbol{E}^2 = \boldsymbol{E} \quad , \quad \boldsymbol{E}^\dagger = \boldsymbol{E} \, . \tag{34}$$

In particular, the physical time evolution operator on the physical subspace simply reduces to

$$\hat{U}_{\rm phys}(t_2, t_1) = \boldsymbol{E} \hat{U}(t_2, t_1) \, \boldsymbol{E} = \sum_{\alpha_{n_0}} |E_{n_0}, \alpha_{n_0} >< E_{n_0}, \alpha_{n_0}| = \boldsymbol{E} \, . \tag{35}$$

Once again, this result expresses the world-line reparametrization gauge invariance fact that quantum physical states are time independent, as befits any physical state of a quantum theory for gravity.

Furthermore, it illustrates the general feature that for reparametrization invariant quantum theories, the physical projector embodies all the physical content of matter coupled quantum gravity. In one dimension, the physics of any such system lies within the subspace of quantum matter states whose energy coincides with the cosmological constant. It certainly is a fascinating issue to determine how this conclusion extends to higher dimensional matter coupled quantum gravity theories.

The above characterization of the quantum physical subspace is also reminiscent of topological quantum field theories[9, 10], namely theories whose reparametrization invariant physical content is solely dependent on the topology of the underlying manifold, with in particular a finite dimensional space of quantum states. One could debate whether the matter coupled quantum gravity systems of this section define the simplest examples of topological quantum field theories, since the considered world-line topology remains trivial, but they certainly provide the simplest examples of quantum gravity systems with a space of quantum physical states which is finite dimensional and such that the cosmological constant is necessarily quantized in a manner dependent on the quantum matter dynamics.

4 Faddeev's Reduced Phase Space Formulation

Given that the above quantization of one-dimensional matter coupled quantum gravity is straightforward and free of any ambiguity, including the complete characterization of its physical sector through the physical projector, it is interesting to confront these results with those that follow from alternative approaches to the quantization of constrained dynamics, which all rely of some method to fix the gauge freedom associated to first-class constraints. Often, such gauge fixing procedures run into Gribov problems[11] of one type or another[8], rendering the physical interpretation difficult since the quantized system is then no longer physically equivalent to the one originally considered. In contradistinction, the previous approach solely based on the physical projector but not on any gauge fixing procedure whatsoever, is guaranteed to be free of any Gribov ambiguity[12], and thus to truly represent the actual quantum dynamics of the original system.

This section considers the application to the previous systems of a quite popular gauge fixing procedure, namely Faddeev's reduced phase space approach[8]. This shall be done explicitly for the spherical harmonic oscillator in N dimensions, defined by

$$H(\vec{q},\vec{p})=\frac{1}{2m}\vec{p}^{\,2}+\frac{1}{2}m\omega^2\vec{q}^{\,2}\ , \tag{36}$$

but once again, the conclusions will be seen to remain valid in general. Two distinct gauge fixing conditions will be considered, both leading to the conclusion that for this general class of systems, Faddeev's approach is unable to provide their physically correct quantization as described in Sect.3.

The reason why this specific choice of matter sector is made, is that the infinitesimal Hamiltonian world-line reparametrizations (25) may then readily be extended to finite transformations, given by

$$\begin{array}{rcl}\vec{q}\,'(t) & = & \vec{q}(t)\,\cos\omega\epsilon(t)+\frac{\vec{p}(t)}{m\omega}\,\sin\omega\epsilon(t)\ ,\\ \vec{p}\,'(t) & = & -m\omega\vec{q}(t)\,\sin\omega\epsilon(t)+\vec{p}(t)\,\cos\omega\epsilon(t)\ ,\\ \lambda'(t) & = & \lambda(t)+\dot{\epsilon}(t)\ ,\end{array} \tag{37}$$

$\epsilon(t)$ being an arbitrary function, possibly subjected to boundary conditions depending on the choice of boundary conditions on the phase space variables $\vec{q}(t)$ and $\vec{p}(t)$. A straightforward substitution into the first-order action $S[\vec{q},\vec{p};\lambda]=\int dt[\dot{\vec{q}}\cdot\vec{p}-\lambda(H-\Lambda)]$ then finds

$$S[\vec{q}\,',\vec{p}\,';\lambda']=S[\vec{q},\vec{p};\lambda]+\int dt\,\frac{d}{dt}\left[-\vec{q}\cdot\vec{p}\sin^2\omega\epsilon+\left(\frac{\vec{p}^{\,2}}{2m\omega}-\frac{1}{2}m\omega\vec{q}^{\,2}\right)\sin\omega\epsilon\cos\omega\epsilon+\Lambda\,\epsilon\right]\ , \tag{38}$$

clearly displaying the gauge invariance of the system under local world-line reparametrizations in phase space. Note that for the present system, these transformations coincide with rotations between the configuration space variables $\vec{q}$ and their conjugate momenta $\vec{p}$, a property related to the fact that energy conservation restricts the dynamics onto an invariant N-dimensional torus in phase space.

The latter remark also shows that when this matter system is coupled to one-dimensional gravity, through[6]

$$H_T=\lambda(t)\left[\frac{1}{2m}\vec{p}^{\,2}+\frac{1}{2}m\omega^2\vec{q}^{\,2}-\Lambda\right]\ , \tag{39}$$

whatever the configuration solving the equations of motion, there always exists a finite gauge transformation $\epsilon(t)$ such that one of the position (or momentum) degrees of freedom vanishes at all times, say $q^1(t)=0$. This remark thus suggests to consider a Faddeev reduced phase space formulation of the system defined for instance by the gauge fixing condition

$$\Omega=q^1=0\ . \tag{40}$$

One of the purposes of such a gauge fixing condition is to single out a specific Lagrange multiplier function, namely a specific world-line parametrization, by requiring this condition to be stable under time evolution. Given the present choice as well as arbitrary values for[7] p_1, this implies $\lambda(t)=0$, namely a singular world-line parametrization such that $\tau(t)=t_i$ at all times! Proceeding nonetheless, one finds that the reduced system is represented by the degrees of freedom (q^i,p_i) with $i=2,3,\cdots,N$ whose Dirac brackets are still given by the canonical brackets $\{q^i(t),p_j(t)\}=\delta_{ij}$, while the constraint $\phi=H-\Lambda=0$ and the gauge fixing condition $\Omega=0$ are solved by

$$q^1=0\quad,\quad p_1=\pm\sqrt{2m\left[\Lambda-\sum_i\left(\frac{p_i^2}{2m}+\frac{1}{2}m\omega^2 q^{i\,2}\right)\right]}\ . \tag{41}$$

Furthermore, given the value $\lambda(t)=0$, the effective Hamiltonian which generates time evolution on reduced phase space vanishes identically, $H_{\rm red}=0$.

Consequently, upon canonical quantization of this reduced phase space formulation of the system, it is clear that no quantization condition whatsoever on the cosmological constant Λ may possibly arise, while at the same time the quantum space of states cannot coincide with the finite dimensional subspace of quantum states of the N-dimensional harmonic oscillator whose energy coincides with a specific and quantized value of the cosmological constant $\Lambda=E_{n_0}=\hbar\omega(n_0+N/2)$, namely the totally symmetric $SU(N)$ representation of dimension $d_{n_0}=(N+n_0-1)!/(n_0!(N-1)!)$, as established in Sect.3. In other words, Faddeev's reduced phase space quantization of the system associated to the gauge fixing condition $q^1=0$ leads to a system whose physics is totally different from that of the

[6] In this discussion, we take Λ to be strictly positive, $\Lambda>0$. A vanishing cosmological constant, $\Lambda=0$, is possible only for the trivial configuration $\vec{q}(t)=\vec{0}$, $\vec{p}(t)=\vec{0}$.

[7] The solution $\lambda(t)=0$ may be avoided only if $p_1(t)=0$ at all times, which in general is inconsistent with an arbitrary choice of initial values at $t=t_i$. It is only if $q^1_i=0$ as well as $p_{1,i}=0$ that the $n=1$ degree of freedom decouples entirely from the dynamics.

original reparametrization invariant system[8], a fact which applies in general whenever a given gauge fixing procedure suffers Gribov problems.

Thinking that this conclusion could possibly be related to the singular world-line parametrization such that $\lambda(t) = 0$ which is associated to the choice $q^1 = 0$, one may consider yet another gauge fixing condition, based on the classical solutions to the Hamiltonian equations of motion. Given initial values $\vec{q}_i = \vec{q}(t_i)$ and $\vec{p}_i = \vec{p}(t_i)$, the general solution is

$$\vec{q}(t) = \vec{q}_i \cos\omega(t - t_i) + \frac{1}{m\omega}\vec{p}_i \sin\omega(t - t_i) \quad , \quad \vec{p}(t) = \vec{p}_i \cos\omega(t - t_i) - m\omega\vec{q}_i \sin\omega(t - t_i) \quad . \qquad (42)$$

For the gravity coupled system, let us thus consider the gauge fixing condition

$$\Omega = q^1 - \left[q_i^1 \cos\omega(t - t_i) + \frac{p_{1,i}}{m\omega} \sin\omega(t - t_i)\right] \quad . \qquad (43)$$

Correspondingly, the world-line parametrization is characterized by the Lagrange multiplier

$$\lambda(t) = \frac{1}{p_1(t)}\left[p_{1,i} \cos\omega(t - t_i) - m\omega q_i^1 \sin\omega(t - t_i)\right] \quad , \qquad (44)$$

while the reduced phase space variables are the coordinates (q^i, p_i) with $i = 2, 3, \cdots, N$ whose Dirac brackets remain canonical[9], with the following solutions to the first-class constraint $\phi = 0$ and gauge fixing condition $\Omega = 0$,

$$q^1(t) = q_i^1 \cos\omega(t - t_i) + \frac{p_{1,i}}{m\omega} \sin\omega(t - t_i) \quad , \quad p_1(t) = \pm\sqrt{2m\left[\Lambda - \sum_i \frac{p_i^2}{2m} - \frac{1}{2}m\omega^2\vec{q}^{\,2}\right]} \quad . \qquad (45)$$

Clearly, even though this gauge fixing is nonsingular given the world-line parametrization with $\tau(t) = t_i + \int_{t_i}^{t} dt'\lambda(t')$, canonical quantization of this formulation of the system cannot recover either its correct quantum physical content as established in Sect.3 with in particular the requirement of a quantized cosmological constant Λ. Once again[7, 8], Faddeev's reduced phase space approach fails for these general classes of one-dimensional reparametrization invariant systems. We are forced to such a conclusion for both examples of gauge fixing conditions above, even though the Faddeev-Popov determinant $\{\Omega, \phi\} = p_1/m$ does not vanish in either case. Contrary to what is often claimed, a nonvanishing Faddeev-Popov determinant is not a sufficient condition for an admissible gauge fixing free of Gribov ambiguities.

It would also be possible to consider these difficulties of Faddeev's reduced phase space approach from the path integral viewpoint, but this issue shall be left aside in this note.

5 Hamiltonian BRST Quantization

Another quite popular approach to constrained dynamics through gauge fixing is the BRST invariant one, namely the BFV-BRST Hamiltonian formalism appropriate to the canonical quantization path

[8]In fact, following arguments similar to those developed in Ref.[8] but which shall not be pursued here, the reason for this lack of equivalence may be traced back to Gribov problems of the first and second types associated to the gauge fixing condition $q^1 = 0$.

[9]The reduced phase space Hamiltonian is not given here.

taken in this note[8]. Here for simplicity, this quantization procedure is applied to the one-dimensional harmonic oscillator, whose quantization is best represented in Fock space through creation and annihilation operators a and $a^\dagger$ such that $[a, a^\dagger] = \mathbb{1}$ and whose quantum Hamiltonian is

$$\hat{H} = \hbar\omega \left[a^\dagger a + \frac{1}{2} \right] \ . \tag{46}$$

Correspondingly, the orthonormalized Fock basis is spanned by the vectors $|n> = (a^\dagger)^n |0> /\sqrt{n!}$ with $< n|m> = \delta_{nm}$, which are $\hat{H}$-eigenstates, $\hat{H}|n> = E_n|n>$, with $E_n = \hbar\omega(n + 1/2)$, $(n = 0, 1, 2, \cdots)$.

In the Hamiltonian BFV-BRST approach for the gravity coupled system, phase space is extended by introducing a canonical momentum p_λ conjugate to the Lagrange multiplier λ, with bracket $\{\lambda, p_\lambda\} = 1$, as well as canonical conjugate pairs of ghost degrees of freedom η^a and $\mathcal{P}_a$ of Grassmann odd parity associated to the first-class constraints $G_a = (p_\lambda, \phi)$ $(a = 1, 2)$ with brackets $\{\eta^a, \mathcal{P}_b\} = -\delta^a_b$. Local gauge invariance is then traded for global BRST invariance generated by the nilpotent BRST charge which for the present system reads

$$Q_B = \eta^1 p_\lambda + \eta^2 \left[\frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 - \Lambda \right] \quad , \quad \{Q_B, Q_B\} = 0 \ . \tag{47}$$

The ghosts η^a carry ghost number $(+1)$ and are real under complex conjugation, whereas the ghosts $\mathcal{P}_a$ carry ghost number (-1) and are pure imaginary under complex conjugation.

Time evolution in this extended phase space is generated by a BRST invariant extension of the gauge invariant first-class Hamiltonian of the system. In our case, the general BRST invariant Hamiltonian is of the form

$$H_{\rm eff} = -\{\Psi, Q_B\} \ , \tag{48}$$

where Ψ is some *a priori* arbitrary function defined over extended phase space, which is pure imaginary under complex conjugation, is of ghost number (-1), and is of odd Grassmann parity. As a matter of fact, gauge fixing of the system is performed through a choice for this function Ψ.

Taking for instance

$$\Psi = F(\lambda)\mathcal{P}_1 + \lambda\mathcal{P}_2 \ , \tag{49}$$

where $F(\lambda)$ is some function of the Lagrange multiplier, leads to the BRST invariant Hamiltonian

$$H_{\rm eff} = \lambda \left[\frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 - \Lambda \right] + F(\lambda) p_\lambda - F'(\lambda)\mathcal{P}_1\eta^1 - \mathcal{P}_2\eta^1 \ , \tag{50}$$

and in turn to BRST invariant equations of motion which include

$$\dot{\lambda} = F(\lambda) \ . \tag{51}$$

Hence, a given choice for the function $F(\lambda)$ and for the initial (or final) value $\lambda_i = \lambda(t_i)$ leads to a specific solution $\lambda(t)$, namely a specific world-line parametrization or gauge fixing of the system.

Choices free of Gribov ambiguities are[8] $F(\lambda) = \alpha\lambda + \beta$ where α, β are arbitrary constant parameters. On the other hand, examples of gauge fixing conditions leading to Gribov problems are[8] $F(\lambda) = \alpha\lambda^2 + \beta\lambda + \delta$, $F(\lambda) = \alpha\lambda^3$ or $F(\lambda) = e^{-\alpha\lambda}$. Even though, by construction, the formulation of the system is BRST and thus gauge invariant whatever the choice for Ψ or $F(\lambda)$, the physics that is being described, albeit gauge invariant, does depend on the gauge fixing condition. It is only when the latter is free of any Gribov problem that an admissible gauge fixing has been achieved, and that a description of the system's dynamics which is equivalent to the original one is recovered. The same statement applies already at the classical level for the solutions to the BRST invariant equations of motion given a choice of BRST invariant boundary conditions.

By considering the BRST transformations of the extended phase space degrees of freedom, one realizes that a general choice of BRST invariant boundary conditions, irrespective of the boundary values for the original configuration space coordinates q^n at $t = t_{i,f}$, is such that

$$p_\lambda(t_{i,f}) = 0 \quad , \quad \mathcal{P}_1(t_{i,f}) = 0 \quad , \quad \eta^2(t_{i,f}) = 0 \ . \tag{52}$$

It is thus also for such a choice of BRST invariant external quantum states that the BRST invariant quantum evolution operator is to be considered, namely that these matrix elements are to represent the gauge invariant time evolution of the quantum system given initial and final $\hat{q}^n$-eigenstate configurations q_i^n and q_f^n.

Canonical quantization of the BFV-BRST extended system is straightforward[8]. In the Grassmann even sector, one has the familiar operator algebras $[a, a^\dagger] = \mathbb{1}$ and $[\hat{\lambda}, \hat{p}_\lambda] = i\hbar$. In the ghost sector, one has the fermionic algebra $\{\hat{c}^a, \hat{b}_b\} = \delta^a_b$, with $\hat{c}^a = \hat{\eta}^a$ and $\hat{b}_a = i\hat{\mathcal{P}}_a/\hbar$, and the properties $\hat{c}^{a\dagger} = \hat{c}^a$ and $\hat{b}_a^\dagger = \hat{b}_a$. The representation of this (b, c) algebra is spanned by a four-dimensional Hilbert space with basis vectors $| \pm\pm >$ such that

$$\begin{array}{llll}
\hat{c}^1|--> = |+->\ , & \hat{c}^1|+-> = 0\ , & \hat{c}^1|-+> = |++>\ , & \hat{c}^1|++> = 0\ , \\
\hat{c}^2|--> = |-+>\ , & \hat{c}^2|+-> = -|++>\ , & \hat{c}^2|-+> = 0\ , & \hat{c}^2|++> = 0\ , \\
\hat{b}_1|--> = 0\ , & \hat{b}_1|+-> = |-->\ , & \hat{b}_1|-+> = 0\ , & \hat{b}_1|++> = |-+>\ , \\
\hat{b}_2|--> = 0\ , & \hat{b}_2|+-> = 0\ , & \hat{b}_2|-+> = |-->\ , & \hat{b}_2|++> = -|+->\ ,
\end{array} \tag{53}$$

and whose inner product is defined through the following nonvanishing matrix elements,

$$<--|++> = -\alpha \quad , \quad <+-|-+> = -\alpha \quad , \quad <-+|+-> = \alpha \quad , \quad <++|--> = \alpha\ , \tag{54}$$

with $\alpha = \pm i$. Finally, the ghost number operator reads

$$\hat{Q}_c = \frac{1}{2}\left[\hat{c}^a\hat{b}_a - \hat{b}_a\hat{c}^a\right]\ , \tag{55}$$

so that $|-->$ is of minimal ghost number (-1), $|+->$ and $|-+>$ are both of vanishing ghost number, and $|++>$ is of maximal ghost number $(+1)$.

As is characteristic of the BRST construction, physical gauge invariant states correspond to BRST invariant states of minimal ghost number, thus[10]

$$\hat{Q}_B|\psi_{\rm phys}, t> = 0 \quad , \quad \hat{Q}_c|\psi_{\rm phys}, t> = -|\psi_{\rm phys}, t>\ , \tag{56}$$

where the nilpotent quantum BRST operator is

$$\hat{Q}_B = \hat{c}^1\hat{p}_\lambda + \hat{c}^2\left[\hbar\omega(a^\dagger a + \frac{1}{2}) - \Lambda\right] \quad , \quad \hat{Q}_B^2 = 0\ . \tag{57}$$

Consequently once again, in order to obtain a quantum theory with physical content, it is necessary that the cosmological constant Λ belongs to the matter energy spectrum, namely in the present case

$$\Lambda = \hbar\omega(n_0 + \frac{1}{2})\ , \tag{58}$$

[10] As we shall see, upon imposing Schrödinger's equation, physical states are also time independent.

for some integer value n_0, a condition which we shall henceforth assume to be met. The general solution for physical states is then of the form

$$|\psi_{\rm phys}, t>= |n_0> \otimes|p_\lambda = 0> \otimes|--> \ \psi_{--,n_0}(t)\ , \tag{59}$$

$\psi_{--,n_0}(t)$ being an arbitrary complex function of time, which turns out to be a constant when solving the Schrödinger equation hereafter. Here, $|p_\lambda>$ denotes the $\hat{p}_\lambda$-eigenstate basis normalized such that $< p_\lambda|p'_\lambda >= \delta(p_\lambda - p'_\lambda)$.

As a matter of fact, it is possible to solve for all BRST invariant states irrespective of their ghost number, $\hat{Q}_B|\psi, t>= 0$, leading to the following general decomposition,

$$|\psi, t>= |n_0> \otimes|p_\lambda = 0> \otimes|--> \ \psi_{--,n_0}(t) \ + \ |n_0> \otimes|p_\lambda = 0> \otimes|-+> \ \psi_{-+,n_0}(t) \ + \ \hat{Q}_B|\chi, t>\ , \tag{60}$$

where $|\chi, t>$ is an arbitrary quantum state. In this form, the BRST cohomology classes are explicit as well, showing that there are only two BRST invariant states that are not BRST-exact, up to the arbitrary components $\psi_{--,n_0}(t)$ and $\psi_{-+,n_0}(t)$. Besides the trivial cohomology class which includes states of ghost numbers 0 and $(+1)$, there are two nontrivial classes, one at ghost number (-1) spanned by $|n_0> \otimes|p_\lambda = 0> \otimes|-->$, and the other of vanishing ghost number spanned by $|n_0> \otimes|p_\lambda = 0> \otimes|-+>$. *A priori*, either of these two nontrivial classes could be put into one-to-one correspondence with the physical states spanned by $|n_0>$ that define the correct quantization of this system as constructed in Sect.3. However, when considering the time evolution of BRST invariant states generated by the Schödinger equation to be discussed below, it turns out that only the BRST nontrivial cohomology class of ghost number (-1) survives, leading back precisely to the characterization of physical states in (59) as being only those BRST invariant states that are of minimal ghost number.

Nonetheless, as we shall see shortly, the presence of the extended phase space degrees of freedom, leading to the $|p_\lambda = 0> \otimes|-->$ component of the physical states, implies an ill defined normalization of BRST invariant matrix elements of the BRST invariant quantum evolution operator, namely *in fine*, of the physical states themselves. For instance, although the physical component $|n_0>$ is properly normalized to unity, that of the BRST invariant physical state (59) itself, whether $\psi_{--,n_0}(t)$ is a pure phase factor or not, is ill defined since the inner product of the state $|p_\lambda = 0>$ with itself is the δ-function $\delta(0)$ while that of the state $|-->$ with itself vanishes, leading to an undetermination of the form $0 \cdot \delta(0)$ for the inner product of physical states (59).

The same issue also arises for BRST invariant matrix elements of the BRST invariant quantum evolution operator of the system. For the class of gauge fixing functions of the form (49), the corresponding BRST invariant quantum Hamiltonian is given by

$$\hat{H}_{\rm eff} = \frac{i}{\hbar}\{\hat{\Psi}, \hat{Q}_B\} = \hat{\lambda}\left[\hbar\omega(a^\dagger a + \frac{1}{2}) - \Lambda\right] + F(\hat{\lambda})\hat{p}_\lambda - i\hbar\hat{c}^1\hat{b}_1 F'(\hat{\lambda}) - i\hbar\hat{c}^1\hat{b}_2\ , \tag{61}$$

for which the Schrödinger equation over the extended space of quantum states reads

$$i\hbar\frac{d}{dt}|\psi, t>= \hat{H}_{\rm eff}|\psi, t>\ . \tag{62}$$

Applied to the two nontrivial BRST cohomology classes in (60), one finds $\psi_{--,n_0}(t) = \psi_{--,n_0}$ to be time independent and $\psi_{-+,n_0}(t) = 0$ to vanish. Hence, it is only the cohomology class of minimal ghost number that also solves the Schrödinger equation, and that may thus qualify as the class of physical states, which, furthermore, are independent of the world-line parametrization, indeed corresponding to the physical states constructed in (59) with a constant $\psi_{--,n_0}(t)$ component. In spite of the normalization problems just mentioned, there thus appears a one-to-one correspondence between the physical states $|n_0>\psi_{--,n_0}$ of Dirac's quantization as discussed in Sect.3 and the BRST invariant physical states of minimal ghost number constructed here. The physical content should thus be equivalent in both approaches, provided however that time evolution in the BRST approach be also consistent with that in Dirac's approach.

To understand how gauge fixing and Gribov problems may interfere with such an equivalence precisely at that level, let us now consider the BRST invariant quantum evolution operator associated to a given choice of gauge fixing function Ψ, namely the operator

$$\hat{U}_{\rm BRST}(t_2,t_1)=e^{-i/\hbar(t_2-t_1)\hat{H}_{\rm eff}}=e^{(t_2-t_1)\{\hat{\Psi},\hat{Q}_B\}/\hbar^2}\ . \qquad (63)$$

Naively, matrix elements of this operator between BRST invariant states should be independent of the gauge fixing function Ψ, since the BRST charge is nilpotent, $\hat{Q}_B^2=0$. Indeed, given two states $|\psi_1>$ and $|\psi_2>$ that are annihilated by $\hat{Q}_B$, it would appear that through a direct expansion of the exponential, one has simply

$$<\psi_2|\hat{U}_{\rm BRST}(t_2,t_1)|\psi_1>=<\psi_2|\psi_1>\ . \qquad (64)$$

However as we have observed, the matrix element on the r.h.s. of this relation is ill defined for nontrivial BRST cohomology classes, so that the l.h.s. requires further specification as to its explicit evaluation through some regularization procedure, thereby possibly implying that even though such matrix elements are indeed BRST invariant, and thus gauge invariant as well, they may not be totally independent of the gauge fixing function Ψ. Once again, it is only for an admissible gauge fixing that the correct physics is represented through these matrix elements[7, 8].

As a first illustration of this fact within the present example, let us consider the zero ghost number BRST invariant states that are associated to the BRST invariant boundary conditions (52), namely

$$\begin{array}{rcl}|\psi_i> &=& |q_i>\otimes|p_\lambda=0>\otimes|-+>=\sum_{n=0}^{\infty}|n>\otimes|p_\lambda=0>\otimes|-+><n|q_i>\ \ ,\\ |\psi_f> &=& |q_f>\otimes|p_\lambda=0>\otimes|-+>=\sum_{n=0}^{\infty}|n>\otimes|p_\lambda=0>\otimes|-+><n|q_f>\ ,\end{array} \qquad (65)$$

where $|q>$ denotes the $\hat{q}$-eigenstate basis normalized such that[11] $<q|q'>=\delta(q-q')$. However, a direct evaluation of the matrix elements

$$<n;p_\lambda=0;-+|\hat{U}_{\rm BRST}(t_2,t_1)|m;p_\lambda=0;-+>\ , \qquad (66)$$

leads to an undetermination of the form $0\cdot\delta(0)$, as explained previously. To avoid this ambiguity, one has to relax momentarily the requirement of BRST invariance and consider rather the matrix elements

$$<n;p_{\lambda,2};-+|\hat{U}_{\rm BRST}(t_2,t_1)|m;p_{\lambda,1};-+>\ . \qquad (67)$$

Being no longer BRST invariant, this expression may now acquire a well defined value which, however, may also depend on the choice for Ψ. Hence even in the BRST invariant limit $p_{\lambda,1},p_{\lambda,2}\to 0$ which might lead to a distribution-like expression, it could be that the final result obtained through this form

[11] The matrix elements $<n|q>$ are thus the harmonic oscillator energy eigenstate wave functions.

of regularization of an otherwise ill defined expression retains some dependency on Ψ, albeit a gauge invariant one. As we shall see, this is indeed the case[7, 8].

The evaluation of the matrix elements (67) is best performed through a path integral representation. The (λ, p_λ) and ghost sectors being common to all one-dimensional reparametrization invariant systems, and thus common to that of the relativistic scalar particle in a Minkowski spacetime, one may borrow the result of the path integral evaluation over these degrees of freedom directly from that latter system[7, 8]. With the following choice of sign for the matrix element $<--|++>=i$, and given the class of gauge fixing functions Ψ in (49), one then finds the exact and explicit result,

$$\lim_{p_{\lambda,1},p_{\lambda,2}\to 0} < n; p_{\lambda,2}; -+|\hat{U}_{\rm BRST}(t_2,t_1)|m;p_{\lambda,1};-+>= i\delta_{n,m}\int_{\mathcal{D}}\frac{d\gamma}{2\pi\hbar}\,e^{-i\gamma/\hbar(\hbar\omega(n+1/2)-\Lambda)}\ , \qquad (68)$$

where the real Teichmüller parameter

$$\gamma=\int_{t_1}^{t_2}dt\,\lambda(t)\ , \qquad (69)$$

the Lagrange multiplier $\lambda(t)$ and the domain of integration $\mathcal{D}$ are constructed as follows. Given the gauge fixing condition

$$\dot{\lambda}=F(\lambda) \qquad (70)$$

associated to (49), and an arbitrary integration constant $\lambda_f=\lambda(t_f)$, consider the corresponding solution $\lambda(t;\lambda_f)$ to this equation as well as its Teichmüller parameter $\gamma(\lambda_f)$. As the integration constant λ_f covers the interval $]-\infty,+\infty[$ once with that orientation, the Teichmüller parameter $\gamma(\lambda_f)$ covers a certain oriented domain $\mathcal{D}$ in the real line. This is the domain over which the final integral in (68) must be performed, including its orientation. This is also how all the extended phase space degrees of freedom contribute to the final matrix element, leading to a dependency on the Lagrange multiplier only through its gauge invariant Teichmüller parameter[12].

Even though the final expression (68) is indeed manifestly BRST and gauge invariant, it does depend on the choice of gauge fixing function Ψ or $F(\lambda)$ in as far as the domain of integration $\mathcal{D}$ depends on that choice. An admissible gauge fixing is such that the domain $\mathcal{D}$ coincides exactly once with the real line, which is the case for example for $F(\lambda)=\alpha\lambda+\beta$. However, if one considers for instance $F(\lambda)=\alpha\lambda^3$ with $\alpha>0$ and $(t_2-t_1)>0$, the domain $\mathcal{D}$ is such that the Teichmüller parameter γ ranges from $-\sqrt{2(t_2-t_1)/\alpha}$ to $\sqrt{2(t_2-t_1)/\alpha}$, clearly displaying the dependency of (68) on the parameter α defining the gauge fixing condition. Nevertheless, in the limit $\alpha\to 0$, one recovers the domain associated to an admissible gauge fixing, as is indeed also the choice $F(\lambda)=0$. Hence, contrary to what is often stated, the BRST invariant matrix elements of the BRST invariant quantum evolution operator are not independent of the gauge fixing conditions. For a gauge fixing procedure suffering Gribov ambiguities, such as the choices $F(\lambda)=\alpha\lambda^3$ or $F(\lambda)=\alpha\lambda^2+\beta\lambda+\delta$ when $\alpha\neq 0$, one does not represent the same quantum dynamics as that of the original system.

Assuming now an admissible gauge fixing choice to have been made, in which case the integral in (68) is over the entire real line, one thus finally has the following evaluation of the relevant BRST invariant matrix elements which represent the quantum propagator for physical states within the BFV-BRST Hamiltonian formalism,

$$< n; p_\lambda=0; -+|\hat{U}_{\rm BRST}(t_2,t_1)|m;p_\lambda=0;-+>= i\delta_{n,m}\delta\left(\hbar\omega(n+\frac{1}{2})-\Lambda\right)\ . \qquad (71)$$

[12]The Teichmüller parameter is indeed invariant under all local reparametrizations $\epsilon(t)$ consistent with the chosen boundary conditions at $t=t_{i,f}$, namely $\epsilon(t_{i,f})=0$.

Hence once again, but clearly only provided the gauge fixing procedure is free of any Gribov ambiguity, one recovers the requirement of a quantized value of the cosmological constant which must belong to the energy spectrum of the matter sector, namely in the present instance for some specific positive integer n_0,

$$\Lambda = \hbar\omega(n_0 + \frac{1}{2}) \ . \tag{72}$$

As a matter of fact, provided a specific choice for $F(\lambda)$ is made at the outset, it is also possible to circumvent the path integral evaluation of the relevant matrix element. For example given the admissible gauge fixing with $F(\lambda) = 0$, as well as the following external states

$$|\psi_i >= \int_{-\infty}^{\infty} dp_\lambda \, |n; p_\lambda; -+> \ \psi_i(p_\lambda) \quad , \quad |\psi_f >= \int_{-\infty}^{\infty} dp_\lambda \, |m; p_\lambda; -+> \ \psi_f(p_\lambda) \ , \tag{73}$$

an explicit calculation of the matrix element $< \psi_f|\hat{U}_{\rm BRST}(t_2, t_1)|\psi_i >$ is readily performed, with the result

$$< \psi_f|e^{-i/\hbar(t_2-t_1)\hat{H}_{\rm eff}}|\psi_i >= i\delta_{n,m}(t_2 - t_1) \int_{-\infty}^{\infty} d\lambda \, \tilde{\psi}_f^*(\lambda) \, e^{-i/\hbar\lambda(t_2-t_1)(\hbar\omega(n+1/2)-\Lambda)} \, \tilde{\psi}_i(\lambda) \ , \tag{74}$$

where

$$\tilde{\psi}_{i,f}(\lambda) = \int_{-\infty}^{\infty} \frac{dp_\lambda}{\sqrt{2\pi\hbar}} \, e^{i/\hbar\,\lambda p_\lambda} \, \psi_{i,f}(p_\lambda) \ . \tag{75}$$

In the limit of BRST invariant external states, namely δ-peaked in the conjugate momentum p_λ with $\psi_{i,f}(p_\lambda) = \delta(p_\lambda)$ and thus non normalizable, one recovers the above expression (68) in the case of an admissible gauge fixing condition, namely with $\mathcal{D}$ being the entire real line,

$$< n; p_\lambda = 0; -+|\hat{U}_{\rm BRST}(t_2, t_1)|m; p_\lambda = 0; -+>= i\delta_{n,m} \int_{-\infty}^{\infty} \frac{d\gamma}{2\pi\hbar} \, e^{-i/\hbar\gamma(\hbar\omega(n+1/2)-\Lambda)} \ , \tag{76}$$

and thus the same final expression and conclusion as in (71). Nevertheless, one sees that whatever the approach, the evaluation of such BRST invariant matrix elements, associated to the quantum propagator of physical states, requires a non-BRST invariant regularization, leading therefore to the possibility of a dependency of gauge invariant quantities on the gauge fixing conditions, albeit in a gauge invariant manner. Gauge fixing leads to a dynamics defined over the space of gauge orbits of the system, hence to a gauge invariant dynamics, but which of these gauge orbits are included and with which multiplicity is dependent on the choice of gauge fixing procedure. The correct physics and dynamics is represented only for an admissible gauge fixing procedure, namely one which selects once and only once each of the gauge orbits.

As a final remark, note that when the cosmological constant takes a quantized valued within the energy spectrum of the matter sector, $\Lambda = E_{n_0}$, the relevant BRST invariant matrix elements are given in terms of the $\delta(E_n - \Lambda)$ distribution with a singular value for the physical sector $n = n_0$, yet another manifestation of the lack of normalizability of physical states within the BRST quantization approach which is due to the extended phase space degrees of freedom. This situation is to be contrasted with the identification of physical states within Dirac's approach, in which everything is so much more straightforward and transparent when expressed in terms of the physical projector.

6 Conclusions

Inspired by a simple example that John once showed me, this note has discussed the coupling of quantum gravity in one dimension to an arbitrary quantum mechanical interacting matter sector. One of the main conclusions of this analysis is that, as a very general result, and even though one-dimensional gravity is quantum dynamics free, the cosmological constant in such systems needs to take a value that belongs to the energy spectrum of the quantum matter sector, and thus to be quantized. This is certainly a very intriguing conclusion, and calls for its possible extension to higher dimensional theories of quantum gravity coupled to interacting quantum matter, eventually shedding some new light onto the cosmological constant problem.

The quantization of these one-dimensional matter coupled quantum gravity systems is most efficient and simple based on John's physical projector approach, lending itself to the immediate identification of the gauge invariant physical sector and of the quantization requirement on the cosmological constant. Such a circumstance enables one to contrast the efficacy of that approach to that of other methods developed over the years towards the quantization of gauge invariant, and more generally, of constrained dynamics.

As has been explicitly illustrated, Faddeev's reduced phase space approach simply fails for one-dimensional reparametrization invariant systems, for reasons that may be traced back to the appearance of Gribov ambiguities whatever the gauge fixing conditions that might be introduced. Even though the resulting formulation remains gauge invariant, it is not independent of the gauge fixing conditions, and thus does not necessarily reproduce the same quantum physics as that of the original system, as is indeed always the case in the presence of Gribov problems. Given that most approaches towards quantum gravity rely one way or another on Faddeev's gauge fixing procedure, the fact that it already fails in one dimension and does not capture the then necessary quantization condition on the cosmological constant leaves totally open the possibility that such a quantization is also required in higher dimensions.

The Hamiltonian BRST invariant quantization method has also been applied to these classes of systems, to explicitly illustrate once again the dependency of the resulting quantum dynamics on the gauge fixing condition. It is only for an admissible gauge fixing procedure that BRST invariant physical states are in one-to-one correspondence with those readily identified within Dirac's approach, and that the requirement of a quantized cosmological constant is recovered. Nevertheless, the introduction of extended degrees of freedom, among which the ghost sector required to balance the gauge variant variables, implies that BRST invariant physical states correspond to non normalizable quantum states, a technical difficulty which is avoided altogether within Dirac's approach and the physical projector.

These general classes of one-dimensional matter coupled quantum gravity models clearly illustrate the various advantages of John's physical projector approach, in which no gauge fixing procedure whatsoever is required and which thus at the outset avoids having to address the difficult and subtle issues of Gribov problems which plague essentially any gauge fixing conditions being used for systems of physical interest. Beginning with the intringuing question regarding the possible quantized value of the cosmological constant, the exploration of the efficacy and the new insights that the physical projector may bring to constrained dynamics thus appears to be of potential great interest to the understanding of the gauge invariant theories on which our present description of the physical world and its fundamental interactions is based.

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