Gravitational radiation from compact binaries in scalar-tensor gravity

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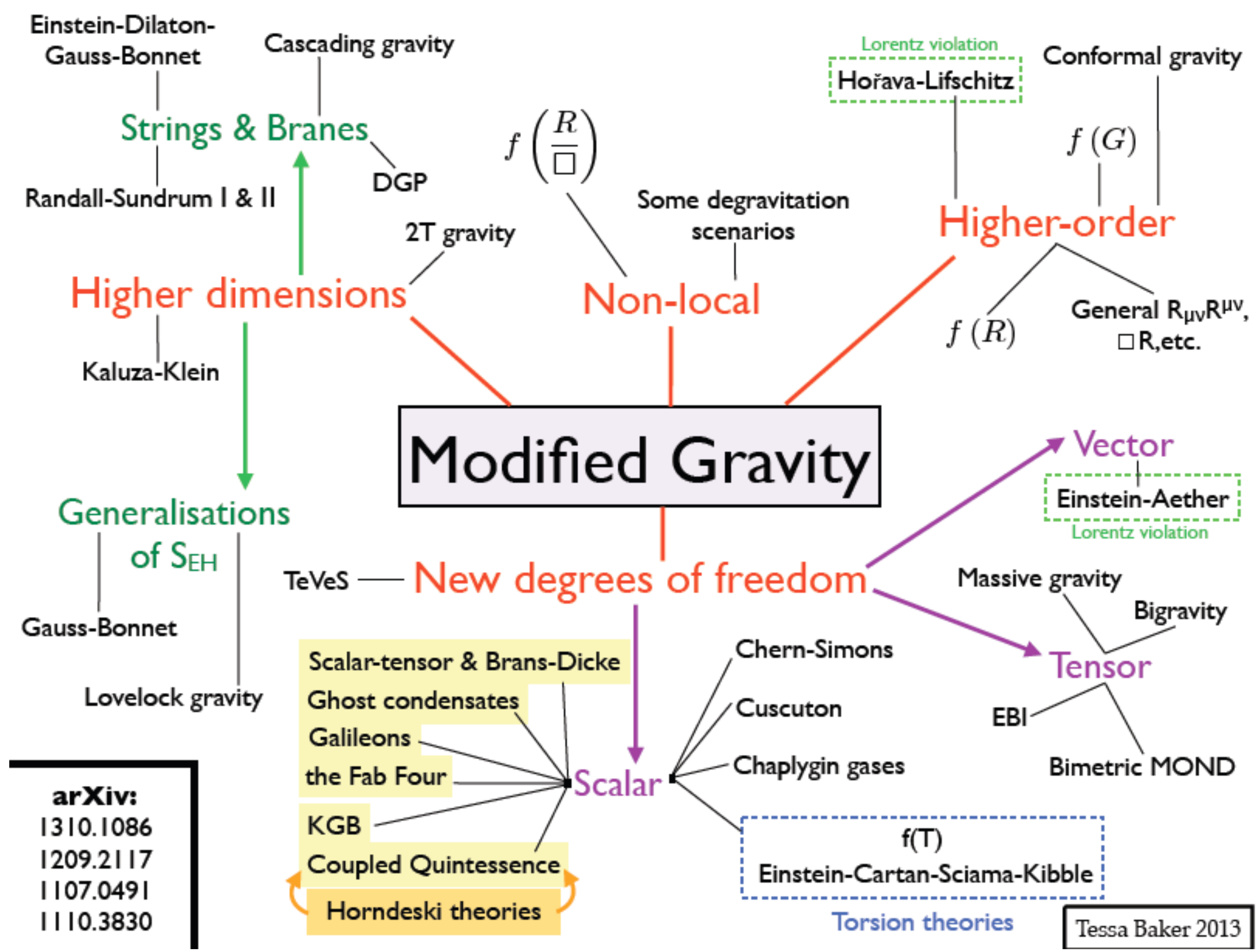
Testing general relativity

- General relativity has withstood every test we have thrown at it for almost 100 years.
- But we know it must fail at some point (no quantum theory of gravity)!
    - When? How?
- Dark matter/dark energy?



Scalar-tensor theories of gravity

- Scalar-tensor (ST) theories are popular alternatives to general relativity (GR).
  - Simple modification
  - Well-motivated:
    - Possible low-energy limit of string theory
    - $f(R)$ theories designed to explain cosmic acceleration can be recast as ST

- Constrained by solar system and binary pulsar tests, but not in strong field (e.g., coalescing compact binaries made of neutron stars and/or black holes).
  - Will be tested by gravitational-wave (GW) detectors like LIGO, LISA
- Goal: Calculate highly accurate (2PN) waveforms for compact binaries in scalar-tensor theories

The starting point: ST action and field equations

$$S = \frac{1}{16\pi} \int \left[\phi R - \frac{\omega(\phi)}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] \sqrt{-g}\ d^4x + S_m(\mathfrak{m}, g_{\mu\nu})$$

$$G_{\mu\nu} = \frac{8\pi}{\phi} T_{\mu\nu} + \frac{\omega(\phi)}{\phi^2} \left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\lambda}\phi^{,\lambda} \right) + \frac{1}{\phi} (\phi_{;\mu\nu} - g_{\mu\nu} \Box_g \phi)$$

$$\Box_g \phi = \frac{1}{3 + 2\omega(\phi)} \left(8\pi T - 16\pi\phi \frac{\partial T}{\partial \phi} - \frac{d\omega}{d\phi} \phi_{,\lambda}\phi^{,\lambda} \right)$$

- We assume no potential/mass for the scalar field.
- Coupling $\omega(\phi)$ is not limited to a constant (i.e., not Brans-Dicke).
- “Jordan frame”: Scalar field does not couple directly to matter.
  - It can couple indirectly: The mass of a compact body depends on its own self-gravity. In ST theories, ϕ controls the effective local value of the gravitational constant.
  - So how do we handle the source term?

Defining the source: Eardley approach

- We can consider the compact bodies (NS, BH) to be point masses. Then the stress-energy tensor is just a sum of delta functions:

$$T^{\mu\nu}(x^\alpha) = (-g)^{-1/2} \sum_A M_A(\phi) u_A^\mu u_A^\nu (u_A^0)^{-1} \delta^3(\mathbf{x} - \mathbf{x}_A)$$

- However, each mass depends on ϕ :

$$M_A(\phi) = M_{A0} + \left(\frac{dM_A}{d\phi}\right)_0 \delta\phi + \frac{1}{2}\left(\frac{d^2 M_A}{d\phi^2}\right)_0 \delta\phi^2 + \frac{1}{6}\left(\frac{d^3 M_A}{d\phi^3}\right)_0 \delta\phi^3 + \cdots$$

- The sensitivity of a body's mass to variations in the scalar field is defined as

$$s_A \equiv \left(\frac{d \ln M_A(\phi)}{d \ln \phi}\right)_0$$

  - Weak field: s_A ~ gravitational energy per unit mass
  - For NS, s_A ~ 0.1-0.3 (depending on EOS). For BH, s_A = 0.5.
  - Also need derivatives (zero for BH): $s'_A \equiv \left(\frac{d^2 \ln M_A(\phi)}{d(\ln \phi)^2}\right)_0$

$$s''_A \equiv \left(\frac{d^3 \ln M_A(\phi)}{d(\ln \phi)^3}\right)_0$$

Rewriting the field equations

- Direct Integration of the Relaxed ~~Einstein~~ Scalar-Tensor equations (Will, Wiseman, and Pati)
- Rescale the scalar field: $\varphi \equiv \dfrac{\phi}{\phi_0}$
- Define a “gravitational field”: $\tilde{h}^{\mu\nu} \equiv \eta^{\mu\nu} - \sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}$ $\qquad \tilde{g}_{\mu\nu} \equiv \varphi g_{\mu\nu}$
- Impose the Lorenz gauge condition: $\tilde{h}^{\mu\nu}{}_{,\nu} = 0$
- This lets us write the field equations as:

$$\Box_\eta \tilde{h}^{\mu\nu} = -16\pi\tau^{\mu\nu} \qquad \tau^{\mu\nu} \equiv (-g)\frac{\varphi}{\phi_0}T^{\mu\nu} + \frac{1}{16\pi}(\Lambda^{\mu\nu} + \Lambda_s^{\mu\nu})$$

$$\Lambda^{\mu\nu} \equiv 16\pi(-\tilde{g})\tilde{t}^{\mu\nu}_{LL} + \tilde{h}^{\mu\alpha}{}_{,\beta}\tilde{h}^{\nu\beta}{}_{,\alpha} - \tilde{h}^{\alpha\beta}\tilde{h}^{\mu\nu}{}_{,\alpha\beta}$$

$$\Lambda_s^{\mu\nu} \equiv \frac{(3+2\omega)}{\varphi^2}\varphi_{,\alpha}\varphi_{,\beta}\left(\tilde{\mathfrak{g}}^{\mu\alpha}\tilde{\mathfrak{g}}^{\nu\beta} - \frac{1}{2}\tilde{\mathfrak{g}}^{\mu\nu}\tilde{\mathfrak{g}}^{\alpha\beta}\right)$$

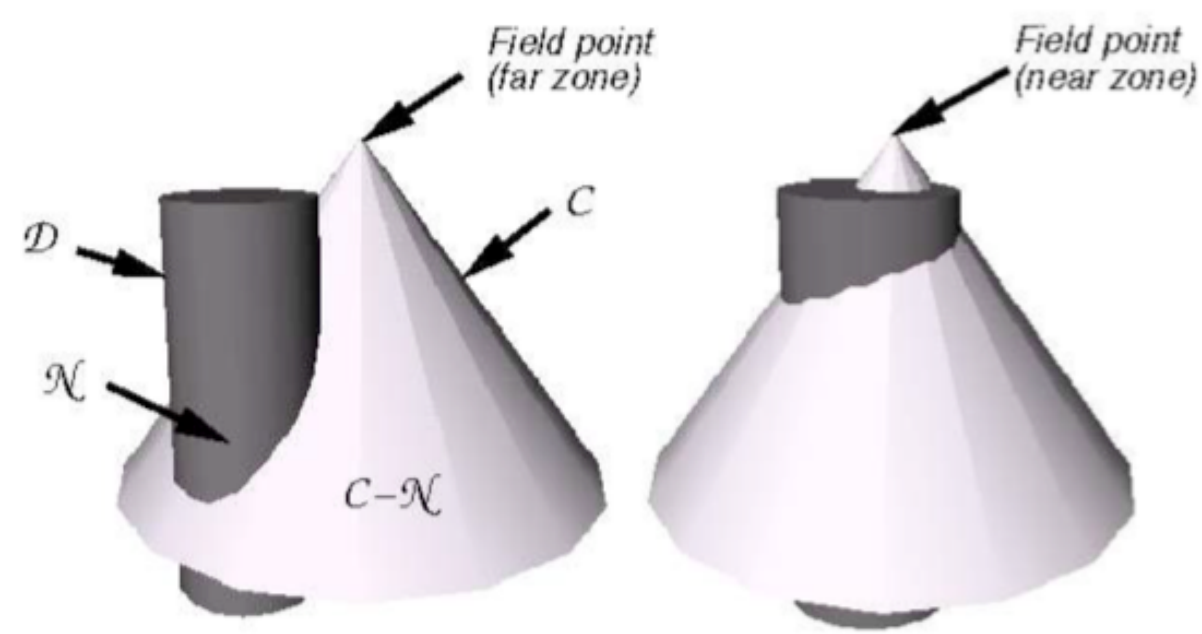
$$\Box_\eta \varphi = -8\pi\tau_s \qquad \tau_s \equiv -\frac{1}{3+2\omega}\sqrt{-g}\frac{\varphi}{\phi_0}\left(T - 2\varphi\frac{\partial T}{\partial\varphi}\right) - \frac{1}{8\pi}\tilde{h}^{\alpha\beta}\varphi_{,\alpha\beta} + \frac{1}{16\pi}\frac{d}{d\varphi}\left[\ln\left(\frac{3+2\omega}{\varphi^2}\right)\right]\varphi_{,\alpha}\varphi_{,\beta}\tilde{\mathfrak{g}}^{\alpha\beta}$$

Solving the wave equations

- Flat-spacetime wave equations can be solved formally using a retarded Green's function:

$$\tilde{h}^{\mu\nu}(t,\mathbf{x}) = 4\int \frac{\tau^{\mu\nu}(t',\mathbf{x}')\delta(t'-t+|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}d^4x' \qquad \varphi(t,\mathbf{x}) = 2\int \frac{\tau_s(t',\mathbf{x}')\delta(t'-t+|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}d^4x'$$

- Divide spacetime into two zones, split at $\mathcal{R} \equiv \lambda\!\!\!^{-} = \lambda/2\pi \sim \mathcal{S}/v$
  - Near zone: $r < \mathcal{R}$ Radiation (far) zone: $r > \mathcal{R}$
- Equations of motion: Evaluate in near zone.
- GWs: Evaluate in radiation zone.
- Near-zone integration: Perform a slow-motion expansion: $\epsilon \sim v^2 \sim m/r$
- Radiation-zone integration: Change variables.
- Ignore $\mathcal{R}$ dependences.



Path to the equations of motion

- Evaluate integrals in near zone.
- Only need to integrate over near zone (to order we need).
- Calculate the near-zone fields iteratively:

$$
\begin{aligned}
\tilde{h}^{00} &\equiv N\,, \\
\tilde{h}^{0i} &\equiv K^i\,, \\
\tilde{h}^{ij} &\equiv B^{ij}\,, \\
\tilde{h}^{ii} &\equiv B\,, \\
\varphi &\equiv 1+\Psi
\end{aligned}
$$

- Calculate the metric:

$$
\begin{aligned}
g_{00} &= -1+\left(\frac{1}{2}N+\Psi\right)+\left(\frac{1}{2}B-\frac{3}{8}N^2-\frac{1}{2}N\Psi-\Psi^2\right) \\
&\quad +\left(\frac{5}{16}N^3-\frac{1}{4}NB+\frac{1}{2}K^jK^j+\frac{3}{8}N^2\Psi-\frac{1}{2}B\Psi+\frac{1}{2}N\Psi^2+\Psi^3\right)+O(\epsilon^4) \\
g_{0i} &= -K^i+\left(\frac{1}{2}N+\Psi\right)K^i+O(\epsilon^{7/2}) \\
g_{ij} &= \delta^{ij}\left\{1+\left(\frac{1}{2}N-\Psi\right)-\left(\frac{1}{8}N^2+\frac{1}{2}B+\frac{1}{2}N\Psi-\Psi^2\right)\right\}+O(\epsilon^3)
\end{aligned}
$$

- Calculate Christoffel symbols.

Equations of motion: results

- Using Bianchi, EOM is $T^{\mu\nu}{}_{;\nu} = \frac{\partial T}{\partial \phi}\phi^{,\mu}$

$$\frac{dv^j}{dt} + \Gamma^j_{\alpha\beta}v^\alpha v^\beta - \Gamma^0_{\alpha\beta}v^\alpha v^\beta v^j = -\frac{1}{M_A(u^0)^2}\frac{dM_A}{d\phi}(\phi^{,j} - \phi^{,0}v^j)$$

- Integrate over bodies, then turn into relative equation of motion. When you do all this, you get...

$$\frac{d^2\mathbf{x}}{dt^2} = -\frac{G\alpha m}{r^2}\mathbf{n} + \frac{G\alpha m}{r^2}\left[\mathbf{n}(A_{PN} + A_{2PN}) + \dot{r}\mathbf{v}(B_{PN} + B_{2PN})\right] \longleftarrow \text{Conservative}$$

$$+\frac{8}{5}\eta\frac{(G\alpha m)^2}{r^3}\left[\dot{r}\mathbf{n}(A_{1.5PN} + A_{2.5PN}) - \mathbf{v}(B_{1.5PN} + B_{2.5PN})\right] \longleftarrow \text{Radiation reaction}$$

- $G \equiv \frac{1}{\phi_0}\frac{4+2\omega_0}{3+2\omega_0}$ is chosen so g_{00} matches GR for a perfect fluid.

$$\alpha \equiv \frac{3+2\omega_0}{4+2\omega_0} + \frac{(1-2s_1)(1-2s_2)}{4+2\omega_0}$$

Calculating the GWs: near-zone integrals

- Near-zone contribution can be written as: $\tilde{h}^{\mu\nu}_{\mathcal{N}}(t,\mathbf{x}) = 4\sum_{q=0}^{\infty}\frac{(-1)^q}{q!}\left(\frac{1}{R}M^{\mu\nu k_1\cdots k_q}\right)_{,k_1\cdots k_q}$

$$M^{\mu\nu k_1\cdots k_q}(\tau) \equiv \int_{\mathcal{M}} \tau^{\mu\nu}(\tau,\mathbf{x}')x'^{k_1}\cdots x'^{k_q}d^3x'$$

- But for GWs, need only the spatial piece, leading term in 1/R expansion:

$$\tilde{h}^{ij}_{\mathcal{N}}(t,\mathbf{x}) = \frac{2}{R}\frac{d^2}{dt^2}\sum_{m=0}^{\infty}\hat{N}^{k_1}\cdots\hat{N}^{k_m}I^{ijk_1\cdots k_m}_{\mathrm{EW}}(\tau)$$

- “Epstein-Wagoner” (EW) moments: $I^{ij}_{\mathrm{EW}} \equiv \int_{\mathcal{M}}\tau^{00}x^{ij}d^3x + I^{ij}_{\mathrm{EW\ (surf)}}$
  - Quadrupole: 0PN+ order, generates 0PN+ GWs
  - Octupole: 0.5PN+ order, generates 0.5PN+ GWs...

- Write as two-body moments in relative coordinates. Take time derivatives using equations of motion.

Calculating the GWs: radiation-zone integrals

- Old 1.5PN piece:

$$\tilde{h}^{ij} = \frac{4G(1-\zeta)m}{R}\left[\frac{11}{12}\dddot{\mathcal{I}}^{ij} + \int_0^\infty ds\, \overset{(4)}{\mathcal{I}^{ij}}(\tau - s)\ln\frac{s}{2R+s}\right]$$

instantaneous   hereditary

$$\mathcal{I}^{ij} = G(1-\zeta)\sum_A m_A x_A^{ij}$$

- This is a gravitational-wave tail.
- Old 2PN piece:

$$\tilde{h}^{ij} = \frac{4G(1-\zeta)m}{R}\left[\frac{97}{180}\overset{(4)}{\mathcal{I}^{ija}}\hat{N}^a + \frac{1}{3}\hat{N}^a\int_0^\infty ds\, \overset{(5)}{\mathcal{I}^{ija}}(\tau - s)\ln\frac{s}{2R+s} - \frac{14}{9}\epsilon^{(i|ab}\dddot{\mathcal{J}}^{b|j)}\hat{N}^a\right.$$
$$\left. - \frac{4}{3}\hat{N}^a\epsilon^{(i|ab}\int_0^\infty ds\, \overset{(4)}{\mathcal{J}^{b|j)}}(\tau - s)\ln\frac{s}{2R+s}\right]$$

Calculating the GWs: radiation-zone integrals

- New 1.5PN piece:

$$\tilde{h}^{ij} = \frac{4G(1-\zeta)m_s}{R}\left(-\frac{1}{12}\dddot{\mathcal{I}}_s^{ij}\right) + \frac{4}{R}\frac{1-\zeta}{\zeta}\left(\frac{1}{6}\int_{-\infty}^{\tau}\ddot{\mathcal{I}}_s^{i}(\tau')\ddot{\mathcal{I}}_s^{j}(\tau')d\tau' - \frac{1}{6}\dot{\mathcal{I}}_s^{(i}\ddot{\mathcal{I}}_s^{j)} - \frac{1}{18}\mathcal{I}_s^{(i}\dddot{\mathcal{I}}_s^{j)}\right)$$

  - Note the hereditary term without a logarithm! It has a DC (zero-frequency) component. This is gravitational-wave memory.
  - GR memory enters at 2.5PN (quadrupole-quadrupole). Here, 1.5PN (dipole-dipole).

- New 2PN piece:

$$\tilde{h}^{ij} = \frac{4G(1-\zeta)m_s}{R}\left(-\frac{1}{60}\overset{(4)}{\mathcal{I}}{}_s^{ija}\hat{N}^a\right) + \frac{4}{R}\frac{1-\zeta}{\zeta}\left[\frac{1}{10}\hat{N}^a\int_{-\infty}^{\tau}d\tau'\ddot{\mathcal{I}}_s^{(a}(\tau')\dddot{\mathcal{I}}_s^{ij)}(\tau') + \left(-\frac{1}{15}\mathcal{I}_s^a\overset{(4)}{\mathcal{I}}{}_s^{ij} - \frac{1}{60}\dot{\mathcal{I}}_s^a\dddot{\mathcal{I}}_s^{ij}\right.\right.$$
$$\left.\left. - \frac{1}{60}\ddot{\mathcal{I}}_s^a\ddot{\mathcal{I}}_s^{ij} + \frac{1}{60}\dddot{\mathcal{I}}_s^a\dot{\mathcal{I}}_s^{ij} - \frac{1}{60}\overset{(4)}{\mathcal{I}}{}_s^a\mathcal{I}_s^{ij} + \frac{1}{30}\mathcal{I}_s^{(i}\overset{(4)}{\mathcal{I}}{}_s^{j)a} - \frac{1}{30}\dot{\mathcal{I}}_s^{(i}\dddot{\mathcal{I}}_s^{j)a} - \frac{1}{30}\ddot{\mathcal{I}}_s^{(i}\ddot{\mathcal{I}}_s^{j)a} - \frac{1}{10}\dddot{\mathcal{I}}_s^{(i}\dot{\mathcal{I}}_s^{j)a}\right)\hat{N}^a\right]$$

  - Non-logarithmic but not DC => not memory

GWs: 0PN, 0.5PN, 1PN

$$\tilde{h}^{ij} = \frac{2G(1-\zeta)\mu}{R}[\tilde{Q}^{ij} + P^{1/2}Q^{ij} + PQ^{ij} + P^{3/2}Q^{ij}_{\mathcal{N}} + P^{3/2}Q^{ij}_{\mathcal{C}-\mathcal{N}} + P^2Q^{ij}_{\mathcal{N}} + P^2Q^{ij}_{\mathcal{C}-\mathcal{N}} + O(\epsilon^{5/2})]_{TT}$$

Can ignore non-TT terms along the way

$$\tilde{Q}^{ij} = 2\left(v^{ij} - \frac{G\alpha m}{r}\hat{n}^{ij}\right)$$

$$P^{1/2}Q^{ij} = \frac{\delta m}{m}\left\{3(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})\frac{G\alpha m}{r}[2\hat{n}^{(i}v^{j)} - \dot{r}\hat{n}^{ij}] + (\hat{\mathbf{N}}\cdot\mathbf{v})\left[\frac{G\alpha m}{r}\hat{n}^{ij} - 2v^{ij}\right]\right\}$$

$$\begin{aligned}
PQ^{ij} = \frac{1}{3}(1-3\eta)\Bigg\{&(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})^2\frac{G\alpha m}{r}\left[\left(3v^2 - 15\dot{r}^2 + 7\frac{G\alpha m}{r}\right)\hat{n}^{ij} + 30\dot{r}\hat{n}^{(i}v^{j)} - 14v^{ij}\right] \\
&+(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})(\hat{\mathbf{N}}\cdot\mathbf{v})\frac{G\alpha m}{r}[12\dot{r}\hat{n}^{ij} - 32\hat{n}^{(i}v^{j)}] + (\hat{\mathbf{N}}\cdot\mathbf{v})^2\left[6v^{ij} - 2\frac{G\alpha m}{r}\hat{n}^{ij}\right]\Bigg\} \\
+\frac{1}{3}\Bigg\{&\left[3(1-3\eta)v^2 - 2(2-3\eta)\frac{G\alpha m}{r}\right]v^{ij} + 4\frac{G\alpha m}{r}\dot{r}(5 + 3\eta + 3\bar{\gamma})\hat{n}^{(i}v^{j)} \\
&+\frac{G\alpha m}{r}\left[3(1-3\eta)\dot{r}^2 - (10 + 3\eta + 6\bar{\gamma})v^2 + \left(29 + 12\bar{\gamma} + 12\bar{\beta}_+ - 12\frac{\delta m}{m}\bar{\beta}_-\right)\frac{G\alpha m}{r}\right]\hat{n}^{ij}\Bigg\}
\end{aligned}$$

GWs: 1.5PN (near-zone)

$$
\begin{aligned}
P^{3/2}Q^{ij}_{\mathcal{N}} = {} & \frac{\delta m}{m}(1-2\eta)\Bigg\{(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})^3\frac{G\alpha m}{r}\Bigg[\frac{5}{4}\left(3v^2-7\dot{r}^2+6\frac{G\alpha m}{r}\right)\dot{r}\hat{n}^{ij}-\frac{17}{2}\dot{r}v^{ij} \\
& -\frac{1}{6}\left(21v^2-105\dot{r}^2+44\frac{G\alpha m}{r}\right)\hat{n}^{(i}v^{j)}\Bigg]+\frac{1}{4}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})^2(\hat{\mathbf{N}}\cdot\mathbf{v})\frac{G\alpha m}{r}\Bigg[58v^{ij} \\
& +\left(45\dot{r}^2-9v^2-28\frac{G\alpha m}{r}\right)\hat{n}^{ij}-108\dot{r}\hat{n}^{(i}v^{j)}\Bigg]+\frac{3}{2}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})(\hat{\mathbf{N}}\cdot\mathbf{v})^2\frac{G\alpha m}{r}[10\hat{n}^{(i}v^{j)}-3\dot{r}\hat{n}^{ij}] \\
& +\frac{1}{2}(\hat{\mathbf{N}}\cdot\mathbf{v})^3\left[\frac{G\alpha m}{r}\hat{n}^{ij}-4v^{ij}\right]\Bigg\} \\
& +\frac{1}{12}\frac{\delta m}{m}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})\frac{G\alpha m}{r}\Bigg\{2\hat{n}^{(i}v^{j)}\Bigg[\dot{r}^2(63+54\eta+36\bar{\gamma})-\frac{G\alpha m}{r}\Bigg(128-36\eta+48\bar{\gamma}+72\bar{\beta}_+ \\
& -72\frac{\delta m}{m}\bar{\beta}_-\Bigg)+v^2(33-18\eta+24\bar{\gamma})\Bigg]+\hat{n}^{ij}\dot{r}\Bigg[\dot{r}^2(15-90\eta)-v^2(63-54\eta+36\bar{\gamma}) \\
& +\frac{G\alpha m}{r}\left(242-24\eta+96\bar{\gamma}+96\bar{\beta}_+-96\frac{\delta m}{m}\bar{\beta}_-\right)\Bigg]-\dot{r}v^{ij}(186+24\eta+96\bar{\gamma})\Bigg\} \\
& +\frac{\delta m}{m}(\hat{\mathbf{N}}\cdot\mathbf{v})\Bigg\{\frac{1}{2}v^{ij}\left[\frac{G\alpha m}{r}(3-8\eta)-2v^2(1-5\eta)\right]-\hat{n}^{(i}v^{j)}\frac{G\alpha m}{r}\dot{r}(7+4\eta+4\bar{\gamma}) \\
& -\hat{n}^{ij}\frac{G\alpha m}{r}\left[\frac{3}{4}(1-2\eta)\dot{r}^2+\frac{1}{3}\left(26-3\eta+12\bar{\gamma}+6\bar{\beta}_+-6\frac{\delta m}{m}\bar{\beta}_-\right)\frac{G\alpha m}{r}-\frac{1}{4}(7-2\eta+4\bar{\gamma})v^2\right]\Bigg\} \\
& +\frac{16}{3}\eta\left(\frac{G\alpha m}{r}\right)^2\zeta\mathcal{S}_-^2\left(\dot{r}\hat{n}^{ij}-\frac{1}{3}\hat{n}^{(i}v^{j)}\right)
\end{aligned}
$$

Unique position dependence

GWs: 1.5PN (radiation zone)

$$P^{3/2}Q^{ij}_{\mathcal{C}-\mathcal{N}} = 4m\int_0^\infty \left\{\frac{G\alpha m}{r^3}\left[\left(3v^2+\frac{G\alpha m}{r}-15\dot{r}^2\right)\hat{n}^{ij}+18\dot{r}\hat{n}^{(i}v^{j)}-4v^{ij}\right]\right\}_{\tau-s}$$
$$\times\left\{G(1-\zeta)\left[\ln\left(\frac{s}{2R+s}\right)+\frac{11}{12}\right]-\frac{1}{12}G\alpha\zeta\left(\mathcal{S}_+ +\frac{\delta m}{m}\mathcal{S}_-\right)\left(\mathcal{S}_+ -\frac{\delta m}{m}\mathcal{S}_-\right)\right\}ds$$
$$+8G\alpha\mu\zeta\mathcal{S}_-^2\int_0^\infty\left\{\frac{G\alpha m}{r^3}\left[\left(-\frac{1}{6}v^2+\frac{1}{9}\frac{G\alpha m}{r}+\frac{5}{6}\dot{r}^2\right)\hat{n}^{ij}-\dot{r}\hat{n}^{(i}v^{j)}+\frac{2}{9}v^{ij}\right]\right\}_{\tau-s}ds$$

GWs: 2PN (near zone)

$$
\begin{aligned}
P^2Q^{ij}_{\mathcal{N}} = \frac{1}{60}(1-5\eta+5\eta^2)\Bigg\{ & 24(\hat{\mathbf{N}}\cdot\mathbf{v})^4\left[5v^{ij}-\frac{G\alpha m}{r}\hat{n}^{ij}\right] \\
& +\frac{G\alpha m}{r}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})^4\left[2\left(175\frac{G\alpha m}{r}-465\dot{r}^2+93v^2\right)v^{ij}+30\dot{r}\left(63\dot{r}^2-50\frac{G\alpha m}{r}-27v^2\right)\hat{n}^{(i}v^{j)}\right. \\
& \left.+\left(1155\frac{G\alpha m}{r}\dot{r}^2-172\left(\frac{G\alpha m}{r}\right)^2-945\dot{r}^4-159\frac{G\alpha m}{r}v^2+630\dot{r}^2v^2-45v^4\right)\hat{n}^{ij}\right] \\
& +24\frac{G\alpha m}{r}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})^3(\hat{\mathbf{N}}\cdot\mathbf{v})\left[87\dot{r}v^{ij}+5\dot{r}\left(14\dot{r}^2-15\frac{G\alpha m}{r}-6v^2\right)\hat{n}^{ij}\right. \\
& \left.+16\left(5\frac{G\alpha m}{r}-10\dot{r}^2+2v^2\right)\hat{n}^{(i}v^{j)}\right]+288\frac{G\alpha m}{r}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})(\hat{\mathbf{N}}\cdot\mathbf{v})^3[\dot{r}\hat{n}^{ij}-4\hat{n}^{(i}v^{j)}] \\
& +24\frac{G\alpha m}{r}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})^2(\hat{\mathbf{N}}\cdot\mathbf{v})^2\left[\left(35\frac{G\alpha m}{r}-45\dot{r}^2+9v^2\right)\hat{n}^{ij}-76v^{ij}+126\dot{r}\hat{n}^{(i}v^{j)}\right]\Bigg\} \\
& +\frac{1}{15}(\hat{\mathbf{N}}\cdot\mathbf{v})^2\Bigg\{\Bigg[5\left(25-78\eta+12\eta^2+4(1-3\eta)\left(3\bar{\gamma}+\bar{\beta}_+-\frac{\delta m}{m}\bar{\beta}_-\right)\right)\frac{G\alpha m}{r} \\
& -(18-65\eta+45\eta^2+10(1-3\eta)\bar{\gamma})v^2+9(1-5\eta+5\eta^2)\dot{r}^2\Bigg]\frac{G\alpha m}{r}\hat{n}^{ij} \\
& +3\left[5(1-9\eta+21\eta^2)v^2-2(4-25\eta+45\eta^2)\frac{G\alpha m}{r}\right]v^{ij} \\
& +18\left[6-15\eta-10\eta^2+\frac{10}{3}(1-3\eta)\bar{\gamma}\right]\frac{G\alpha m}{r}\dot{r}\hat{n}^{(i}v^{j)}\Bigg\} \\
& +\frac{1}{15}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})(\hat{\mathbf{N}}\cdot\mathbf{v})\frac{G\alpha m}{r}\Bigg\{\Bigg[3(36-145\eta+150\eta^2+20(1-3\eta)\bar{\gamma})v^2 \\
& -5\left(127-392\eta+36\eta^2+56(1-3\eta)\bar{\gamma}+32(1-3\eta)\left(\bar{\beta}_+-\frac{\delta m}{m}\bar{\beta}_-\right)\right)\frac{G\alpha m}{r} \\
& -15(2-15\eta+30\eta^2)\dot{r}^2\Bigg]\dot{r}\hat{n}^{ij}+6[98-295\eta-30\eta^2+50(1-3\eta)\bar{\gamma}]\dot{r}v^{ij} \\
& +2\Bigg[5\left(66-221\eta+96\eta^2+26(1-3\eta)\bar{\gamma}+32(1-3\eta)\left(\bar{\beta}_+-\frac{\delta m}{m}\bar{\beta}_-\right)\right)\frac{G\alpha m}{r} \\
& -9(18-45\eta-40\eta^2+10(1-3\eta)\bar{\gamma})\dot{r}^2-(66-265\eta+360\eta^2+50(1-3\eta)\bar{\gamma})v^2\Bigg]\hat{n}^{(i}v^{j)}\Bigg\}
\end{aligned}
$$

GWs: 2PN (near zone)

$$
\begin{aligned}
&+\frac{1}{60}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})^2\frac{G\alpha m}{r}\Bigg\{\Bigg[3(33-130\eta+150\eta^2+20(1-3\eta)\bar{\gamma})v^4+105(1-10\eta+30\eta^2)\dot{r}^4\\
&\quad+15\left(181-572\eta+84\eta^2+72(1-3\eta)\bar{\gamma}+64(1-3\eta)\left(\bar{\beta}_+-\frac{\delta m}{m}\bar{\beta}_-\right)\right)\frac{G\alpha m}{r}\dot{r}^2\\
&\quad-\left(131-770\eta+930\eta^2-80(1-3\eta)\bar{\gamma}+160(1-3\eta)\left(\bar{\beta}_+-\frac{\delta m}{m}\bar{\beta}_-\right)\right)\frac{G\alpha m}{r}v^2\\
&\quad-60(9-40\eta+60\eta^2+5(1-3\eta)\bar{\gamma})v^2\dot{r}^2\\
&\quad-8\left(131-390\eta+30\eta^2+60(1-3\eta)\bar{\gamma}+65(1-3\eta)\left(\bar{\beta}_+-\frac{\delta m}{m}\bar{\beta}_-\right)\right)\left(\frac{G\alpha m}{r}\right)^2\Bigg]\hat{n}^{ij}\\
&\quad+4\Bigg[(12+5\eta-315\eta^2-10(1-3\eta)\bar{\gamma})v^2-9(39-115\eta-35\eta^2+20(1-3\eta)\bar{\gamma})\dot{r}^2\\
&\quad+5\left(29-104\eta+84\eta^2+8(1-3\eta)\bar{\gamma}+28(1-3\eta)\left(\bar{\beta}_+-\frac{\delta m}{m}\bar{\beta}_-\right)\right)\frac{G\alpha m}{r}\Bigg]v^{ij}\\
&\quad+4\Bigg[15(18-40\eta-75\eta^2+10(1-3\eta)\bar{\gamma})\dot{r}^2\\
&\quad-5\left(197-640\eta+180\eta^2+76(1-3\eta)\bar{\gamma}+80(1-3\eta)\left(\bar{\beta}_+-\frac{\delta m}{m}\bar{\beta}_-\right)\right)\frac{G\alpha m}{r}\\
&+\frac{1}{15}\eta\left(\frac{G\alpha m}{r}\right)^2\zeta\mathcal{S}_-\Bigg\{(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}})\Bigg[192\left(\mathcal{S}_+-\frac{\delta m}{m}\mathcal{S}_-\right)\dot{r}\hat{n}^{(i}v^{j)}\\
&\quad+\left(-120\left(\mathcal{S}_+-\frac{\delta m}{m}\mathcal{S}_-\right)\dot{r}^2+24\left(\mathcal{S}_+-\frac{\delta m}{m}\mathcal{S}_-\right)v^2+4\left(11\mathcal{S}_++19\frac{\delta m}{m}\mathcal{S}_-\right)\frac{G\alpha m}{r}\right)\hat{n}^{ij}\\
&\quad-4\left(17\mathcal{S}_+-7\frac{\delta m}{m}\mathcal{S}_-\right)v^{ij}\Bigg]+(\hat{\mathbf{N}}\cdot\mathbf{v})\left[-16\left(8\mathcal{S}_++7\frac{\delta m}{m}\mathcal{S}_-\right)\hat{n}^{(i}v^{j)}+12\left(7\mathcal{S}_++3\frac{\delta m}{m}\mathcal{S}_-\right)\dot{r}\hat{n}^{ij}\right]\Bigg\}
\end{aligned}
$$

GWs: 2PN (near zone)

$$
\begin{aligned}
&+\frac{1}{60}\Bigg\{\Bigg[\Bigg(467+780\eta-120\eta^2+120(2+3\eta)\bar{\gamma}+10\bar{\gamma}^2+40\left(\bar{\delta}_++\frac{\delta m}{m}\bar{\delta}_-\right)\\
&\quad-40(1-3\eta)\left(\bar{\beta}_+-\frac{\delta m}{m}\bar{\beta}_-\right)\Bigg)\frac{G\alpha m}{r}v^2-15\Bigg(61-96\eta+48\eta^2+\frac{8}{3}(7-12\eta)\bar{\gamma}-\frac{4}{3}\bar{\gamma}^2\\
&\quad-\frac{16}{3}\left(\bar{\delta}_++\frac{\delta m}{m}\bar{\delta}_-\right)+\frac{32}{3}(2-3\eta)\left(\bar{\beta}_+-\frac{\delta m}{m}\bar{\beta}_-\right)\Bigg)\frac{G\alpha m}{r}\dot{r}^2-(144-265\eta-135\eta^2+20(4-9\eta)\bar{\gamma})v^4\\
&\quad+6(24-95\eta+75\eta^2+10(1-3\eta)\bar{\gamma})v^2\dot{r}^2-2\Bigg(642+545\eta+20(29+6\eta)\bar{\gamma}+15(9-2\eta)\bar{\gamma}^2\\
&\quad+80(8+6\eta+3\bar{\gamma})\bar{\beta}_+-80(8+3\bar{\gamma})\frac{\delta m}{m}\bar{\beta}_-+60(1-2\eta)(\bar{\delta}_+-2\bar{\chi}_+)+60\frac{\delta m}{m}(\bar{\delta}_-+2\bar{\chi}_+)\\
&\quad-1440\eta\frac{\bar{\beta}_1\bar{\beta}_2}{\bar{\gamma}}\Bigg)\left(\frac{G\alpha m}{r}\right)^2-45(1-5\eta+5\eta^2)\dot{r}^4\Bigg]\frac{G\alpha m}{r}\hat{n}^{ij}\\
&\quad+\Bigg[4(69+10\eta-135\eta^2+10(4-3\eta)\bar{\gamma})\frac{G\alpha m}{r}v^2-12(3+60\eta+25\eta^2+40\eta\bar{\gamma})\frac{G\alpha m}{r}\dot{r}^2\\
&\quad+45(1-7\eta+13\eta^2)v^4-10\Bigg(56+165\eta-12\eta^2+4(7+24\eta)\bar{\gamma}+\bar{\gamma}^2\\
&\quad+4\left(\bar{\delta}_++\frac{\delta m}{m}\bar{\delta}_-\right)-4(1+3\eta)\bar{\beta}_++4(1-3\eta)\frac{\delta m}{m}\bar{\beta}_-\Bigg)\left(\frac{G\alpha m}{r}\right)^2\Bigg]v^{ij}\\
&\quad+4\Bigg[2(36+5\eta-75\eta^2+5(4-3\eta)\bar{\gamma})v^2-6(7-15\eta-15\eta^2+5(1-3\eta)\bar{\gamma})\dot{r}^2\\
&\quad+5\Bigg(35+45\eta+36\eta^2+8(1+6\eta)\bar{\gamma}-\bar{\gamma}^2+16(1-3\eta)\bar{\beta}_+-8(2-3\eta)\frac{\delta m}{m}\bar{\beta}_-\\
&\quad-4\left(\bar{\delta}_++\frac{\delta m}{m}\bar{\delta}_-\right)\Bigg)\frac{G\alpha m}{r}\Bigg]\frac{G\alpha m}{r}\dot{r}\hat{n}^{(i}v^{j)}\Bigg\}
\end{aligned}
$$

GWs: 2PN (radiation zone)

$$
\begin{aligned}
P^2 Q^{ij}_{\mathcal{C}-\mathcal{N}} = 2m \int_0^\infty &\left\{ \frac{G\alpha m}{r^3} \left[15 \left(3v^2 + 2\frac{G\alpha m}{r} - 7\dot{r}^2 \right) \dot{r}\hat{n}^{ij}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}}) \right.\right. \\
&- \left(13v^2 + \frac{22}{3}\frac{G\alpha m}{r} - 65\dot{r}^2 \right) \left(\hat{n}^{ij}(\hat{\mathbf{N}}\cdot\mathbf{v}) + 2\hat{n}^{(i}v^{j)}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}}) \right) - 40\dot{r}\left(v^{ij}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}}) + 2\hat{n}^{(i}v^{j)}(\hat{\mathbf{N}}\cdot\mathbf{v}) \right) \\
&\left.\left. + 20 v^{ij}(\hat{\mathbf{N}}\cdot\mathbf{v}) \right] \right\}_{\tau-s} \\
&\times \left\{ G(1-\zeta)\frac{\delta m}{m} \ln\left[\left(\frac{s}{2R+s} \right) + \frac{97}{60} \right] - \frac{1}{20} G\alpha\zeta \left(\mathcal{S}_+ + \frac{\delta m}{m}\mathcal{S}_- \right) \left(\frac{\delta m}{m}\mathcal{S}_+ - (1-2\eta)\mathcal{S}_- \right) \right\} ds \\
+ 8G(1-\zeta)\delta m \int_0^\infty &\left\{ \frac{G\alpha m}{r^3} \left[\left(v^2 - \frac{2}{3}\frac{G\alpha m}{r} - 5\dot{r}^2 \right) \left(\hat{n}^{ij}(\hat{\mathbf{N}}\cdot\mathbf{v}) - \hat{n}^{(i}v^{j)}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}}) \right) \right.\right. \\
&\left.\left. - 2\dot{r}\left(v^{ij}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}}) - \hat{n}^{(i}v^{j)}(\hat{\mathbf{N}}\cdot\mathbf{v}) \right) \right] \right\}_{\tau-s} \left[\ln\left(\frac{s}{2R+s} \right) + \frac{7}{6} \right] ds \\
+ \frac{1}{15} G\alpha\mu\zeta\mathcal{S}_- \left(\mathcal{S}_+ - \frac{\delta m}{m}\mathcal{S}_- \right) \int_0^\infty &\left\{ \frac{G\alpha m}{r^3} \left[\left(225v^2 + 18\frac{G\alpha m}{r} - 525\dot{r}^2 \right) \dot{r}\hat{n}^{ij}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}}) \right.\right. \\
&+ \left(-9v^2 - 6\frac{G\alpha m}{r} + 45\dot{r}^2 \right) \hat{n}^{ij}(\hat{\mathbf{N}}\cdot\mathbf{v}) + \left(-162v^2 + 44\frac{G\alpha m}{r} + 810\dot{r}^2 \right) \hat{n}^{(i}v^{j)}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}}) \\
&\left.\left. - 144\dot{r}\hat{n}^{(i}v^{j)}(\hat{\mathbf{N}}\cdot\mathbf{v}) - 276\dot{r}v^{ij}(\hat{\mathbf{N}}\cdot\hat{\mathbf{n}}) + 56v^{ij}(\hat{\mathbf{N}}\cdot\mathbf{v}) \right] \right\}_{\tau-s} ds
\end{aligned}
$$

Parameter set

- Ultimately, everything depends on parameters of the theory ($\omega(\phi)$ and ϕ_0) and the sensitivities of the sources...as well as all of their derivatives!

Parameter	Definition	Parameter	Definition
Scalar-tensor parameters		**Equation of motion parameters**	
G	$\phi_0^{-1}(4+2\omega_0)/(3+2\omega_0)$	**Newtonian**	
ζ	$1/(4+2\omega_0)$	α	$1-\zeta+\zeta(1-2s_1)(1-2s_2)$
λ_1	$(d\omega/d\varphi)_0\zeta^2/(1-\zeta)$	**post-Newtonian**	
λ_2	$(d^2\omega/d\varphi^2)_0\zeta^3/(1-\zeta)$	$\bar{\gamma}$	$-2\alpha^{-1}\zeta(1-2s_1)(1-2s_2)$
Sensitivities		$\bar{\beta}_1$	$\alpha^{-2}\zeta(1-2s_2)^2\left(\lambda_1(1-2s_1)+2\zeta s_1'\right)$
s_A	$[d\ln M_A(\phi)/d\ln\phi]_0$	$\bar{\beta}_2$	$\alpha^{-2}\zeta(1-2s_1)^2\left(\lambda_1(1-2s_2)+2\zeta s_2'\right)$
s_A'	$[d^2\ln M_A(\phi)/d\ln\phi^2]_0$	**2nd post-Newtonian**	
s_A''	$[d^3\ln M_A(\phi)/d\ln\phi^3]_0$	$\bar{\delta}_1$	$\alpha^{-2}\zeta(1-\zeta)(1-2s_1)^2$
		$\bar{\delta}_2$	$\alpha^{-2}\zeta(1-\zeta)(1-2s_2)^2$
		$\bar{\chi}_1$	$\alpha^{-3}\zeta(1-2s_2)^3\left[(\lambda_2-4\lambda_1^2+\zeta\lambda_1)(1-2s_1)-6\zeta\lambda_1 s_1'+2\zeta^2 s_1''\right]$
		$\bar{\chi}_2$	$\alpha^{-3}\zeta(1-2s_1)^3\left[(\lambda_2-4\lambda_1^2+\zeta\lambda_1)(1-2s_2)-6\zeta\lambda_1 s_2'+2\zeta^2 s_2''\right]$

$$\mathcal{S}_+ \equiv \alpha^{-1/2}(1-s_1-s_2)$$
$$\mathcal{S}_- \equiv \alpha^{-1/2}(s_2-s_1)$$

Key facts

- BH binaries indistinguishable from GR (with mass rescaling)
    - Hawking: Single BHs have no scalar hair.
        - Now seen for binaries to 2.5PN order (motion), 2PN (tensor waves)
        - Conjecture: True to all PN orders (with no scalar potential, no matter, ϕ_0 constant). Supported by NR results (Healy et al.), EMRI studies (Yunes et al.)

- BH-NS binaries indistinguishable from GR up to 1PN order, with deviations at higher order depending only on a single parameter $Q \equiv \zeta(1-\zeta)^{-1}(1-2s_1)^2$
    - This may be an interesting parameter to try and measure.
    - But Q is the same in Brans-Dicke theory and generic ST theory. So we can't tell them apart (aside from effects on NS itself!)

The future

- Scalar waves
- Energy flux
- Ready-to-use templates, parameter estimation, etc.

- Trouble: monopole and dipole moments at lower PN order than quadrupole moment (“0PN”). Monopole = -1PN, dipole = -0.5PN
  - For 2PN scalar waves, need monopole moment to 3PN, dipole moment to 2.5PN

$$\varphi_{\mathcal{N}}(t,\mathbf{x}) = \frac{2}{R}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m}{\partial t^m}\int_{\mathcal{M}}\tau_s(u,\mathbf{x}')(\hat{\mathbf{N}}\cdot\mathbf{x}')^m d^3x'$$

- For 2PN energy flux, need 3PN dipole moment and 3PN equations of motion!!!

- For more:
  - Mirshekari and Will PRD 87, 084070 (2013), arXiv:1301.4680
  - Lang PRD 89, 084014 (2014), arXiv: 1310.3320