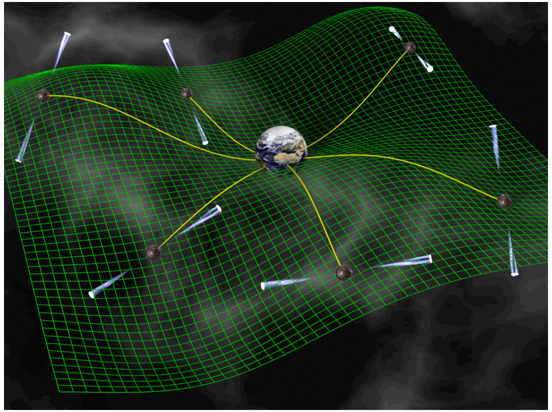
The sensitivity of pulsar timing arrays



Christopher Moore
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Work done in collaboration with Steve Taylor (IoA ➢ JPL) and Dr. Jonathan Gair (IoA)

Outline

- Introduction to pulsar timing arrays (PTAs)
- Sources of gravitational waves for PTA
- Defining a sensitivity curve: frequentist and Bayesian approaches
- Sensitivity of our PTA to different types of source
- Discussion and conclusions

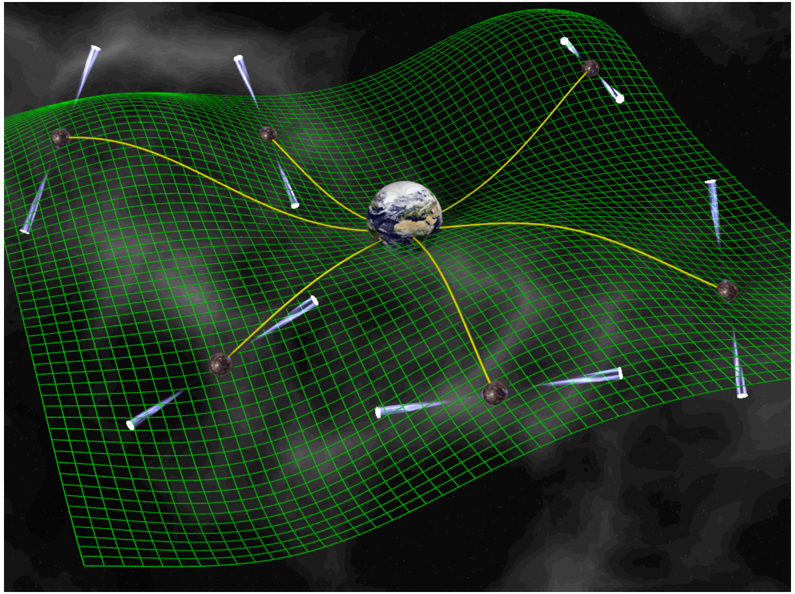
Introduction

An array of galactic millisecond pulsars with extremely high rotational stability

Lines joining them to Earth form the “arms of an interferometer”

A GW between the Earth and a pulsar causes a shift in the pulsar frequency

Using the array of pulsars allows us to cross-correlate the data and exploit the fact that GWs influence all pulsars



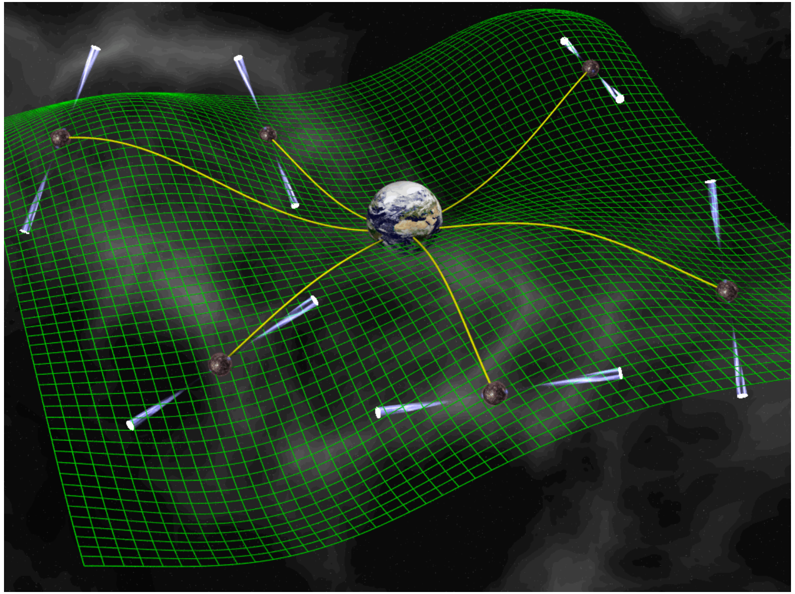
Introduction

Why a PTA is NOT like an interferometer?

The arms are much longer than the light has had time to travel

Very different noise properties! Unlike ground or space-based detectors there's no instrinsic limiting low frequency noise

$$f_{\min} \approx 1/T$$



Sources of GWs

$$1/T \approx 10^{-8}\,\mathrm{Hz} \approx 10^{13}\,M_\odot$$

Or a lower mass binary in the earlier stages of inspiral.

The main source of GWs in this frequency band is thought to be a population of supermassive black hole binaries, with typical masses $\left(10^8\text{-}10^{10}\right) M_\odot$ and redshifts $z \lesssim 2$, in the early inspiral regime of their coalescence

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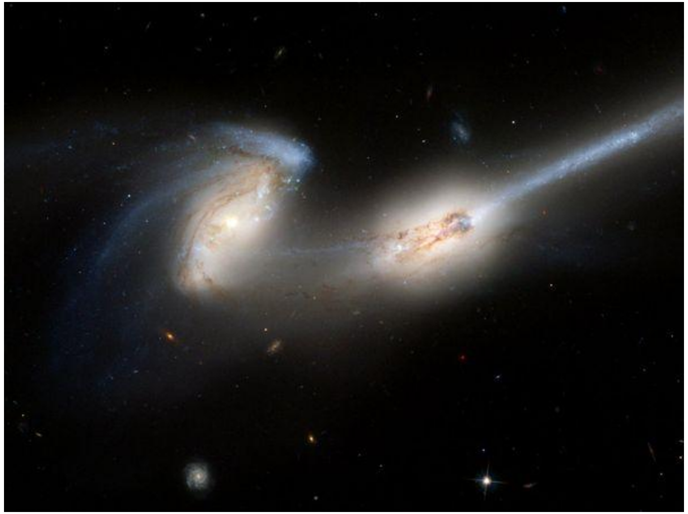
As you move to lower frequencies there are more binaries per frequency bin and they superpose to form a stochastic background

$$\left\langle \tilde{h}_x(f)\tilde{h}_y^*(f')\right\rangle = \frac{1}{2}\delta(f-f')\Gamma_{xy}S_h(f)$$

Sources of GWs

So will we see a stochastic background or individual sources?



Depends of the distribution in amplitude and frequency (or equivalently chirp mass and redshift) of the sources.

Our canonical PTA

- 36 pulsars located randomly on the sky
- All timed to a precision of 100 ns
- Timed fortnightly, i.e. a cadence of 1 / (2 weeks)
- Timed over a total baseline of 5 years

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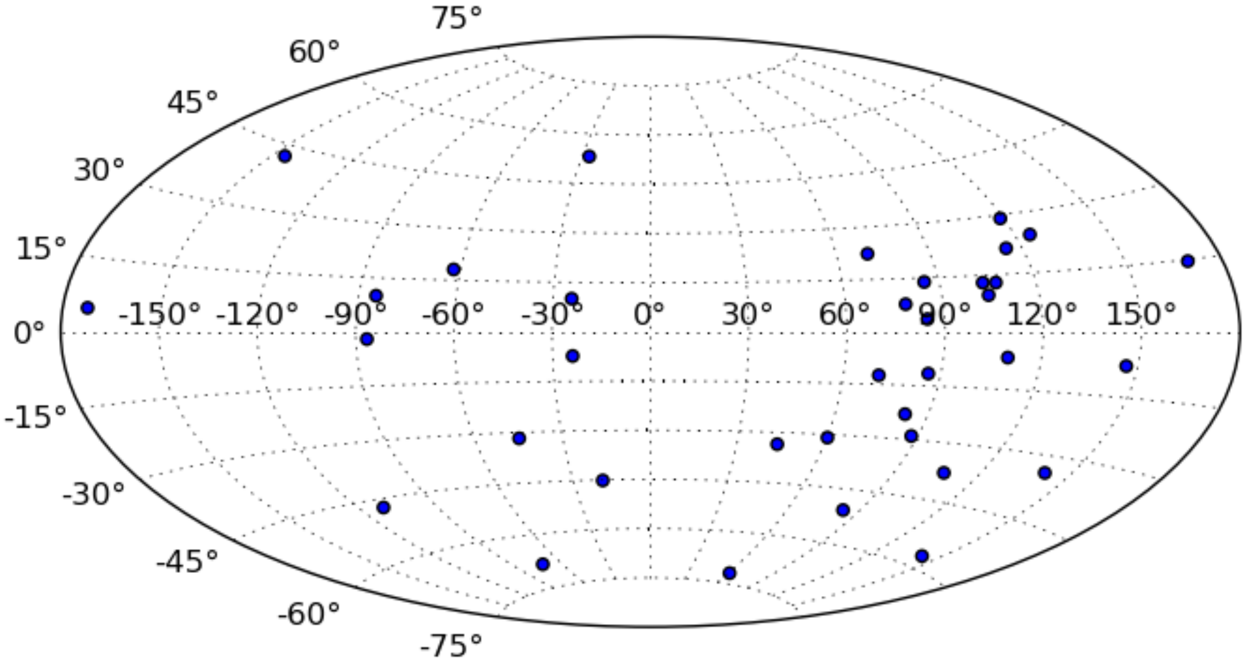
Roughly equivalent to `Open1` of the IPTA mock data challenge

`http://www.ipta4gw.org/?page_id=89`

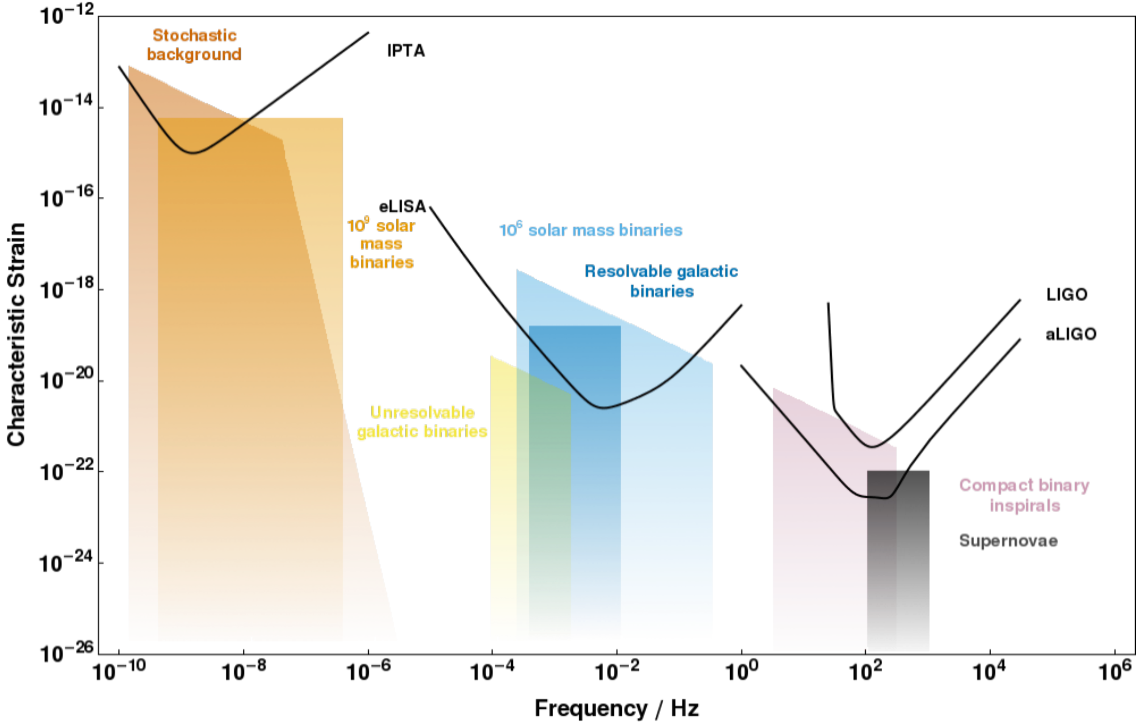
IPTA pulsar locations

J0030+0451
J0437-4715
J0621+1002
J0751+1807
J1012+5307
J1024-0719
J1455-3330
J1603-7202
J1643-1224
J1730-2304
J1738+0333
J1744-1134
J1853+1303
J1909-3744
J1918-0642
J1955+2908
J2124-3358
J2145-0750

J0218+4232
J0613-0200
J0711-6830
J0900-3144
J1022+1001
J1045-4509
J1600-3053
J1640+2224
J1713+0747
J1732-5049
J1741+1351
J1751-2857
J1857+0943
J1910+1256
J1939+2134
J2019+2425
J2129-5721
J2317+1439

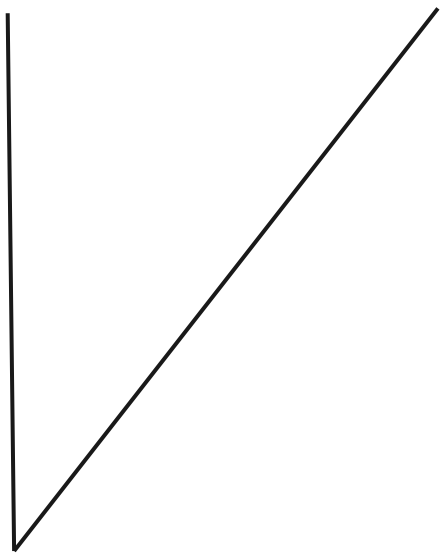


Sensitivity curves

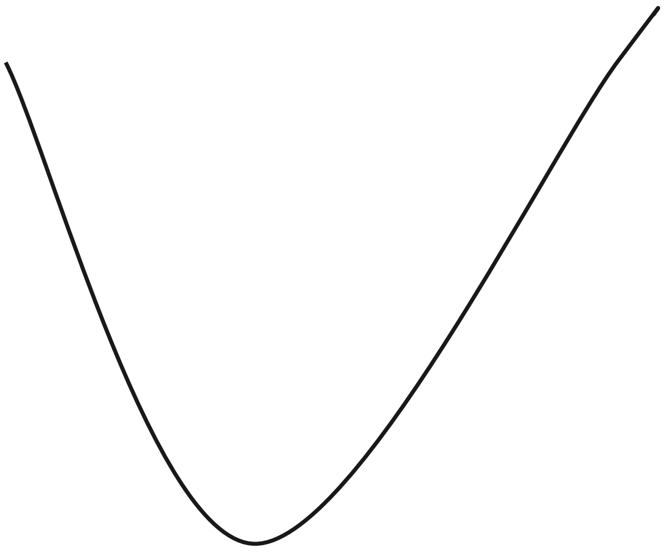


Interactive version of figure online at http://www.ast.cam.ac.uk/~rhc26/sources/

Sensitivity curves



?



Sensitivity curves

Frequentist Approach: Define a detection statistic. The SNR is given by

$$\varrho^2 = \frac{\mu^2}{\sigma^2} = \frac{\langle \mathcal{S} \rangle^2_{s=h+n}}{\langle \mathcal{S}^2 \rangle_{s=n} - \langle \mathcal{S} \rangle^2_{s=n}}$$

We say a source is marginally detectable when the expected value of the SNR is some threshold value, $\rho = \rho_{\rm th}$.

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We say a source is marginally detectable when the expected value of the SNR is some threshold value, $\rho = \rho_{\rm th}$.

Bayesian Approach: Calculate the evidence for two competing hypotheses, the noise and signal hypothesis. The Bayes' factor is the ratio

$$\mathcal{B} \equiv \frac{O_h}{O_n} \quad \text{where } O_i = \int \mathrm{d}\vec{\lambda}\ \Pi\left(\vec{\lambda}\right) \mathcal{L}_i\left(\vec{\lambda}\right)$$

We say a source is marginally detectable when the expected Bayes' factor is some threshold value, $\mathcal{B} = \mathcal{B}_{\rm th}$.

Monochromatic Source

Frequentist approach:

$$\mathbf{s}(t) = \mathbf{n}(t) + \mathbf{h}(t) \;, \quad \text{where } \mathbf{s}(t)^T = \left(s_1(t), s_2(t), \ldots s_{N_p}(t)\right)$$

$$\left\langle \tilde{n}_x(f) \tilde{n}_y^*(f') \right\rangle \; = \; (1/2)\delta(f \; - \; f')\delta_{xy} S_n$$

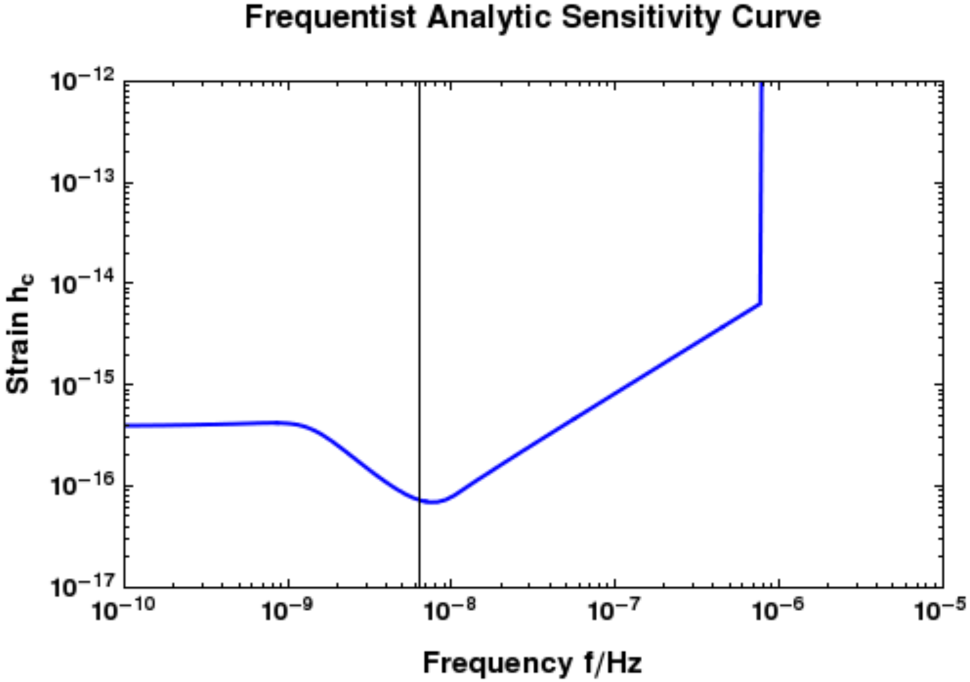
$$\mathcal{S} \; = \int \mathrm{d}t \int \mathrm{d}t' \, \mathbf{s}(t)^T \mathbf{K}^\dagger (t - t') \mathbf{s}(t') = \int \mathrm{d}f \; \tilde{\mathbf{s}}(f)^T \tilde{\mathbf{K}}^\dagger (f) \tilde{\mathbf{s}}(f)$$

$$\varrho^2 = \sum_y \sum_{x>y} \frac{1}{T} \int \mathrm{d}f \; \frac{\left|\tilde{h}_x(f)\right|^2 \left|\tilde{h}_y(f)\right|^2}{S_n^2}$$

$$\varrho^2 = \frac{1}{2} N_p \left(N_p - 1\right) \frac{\chi^4 h_c^4}{T} \int_{1/T}^{1/\delta t} \mathrm{d}f \; \frac{\delta_T^4 (f - f_0)}{f^4 S_n^2}$$

Monochromatic Source

Frequentist approach:



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Frequentist approach (time domain):

$$\varrho^2 \approx \frac{1}{2} N_p \left(N_p - 1\right) T \int_0^T \mathrm{d}t \; \frac{\chi^4 h_c^4 \sin^4\left(2\pi f t + \phi\right)}{\sigma^4 f^4 \delta t^2}$$

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Tends to a power law in the high frequency limit

$$h_c^{\mathrm{HIGH}}(f) \approx \left(\frac{16 \varrho^2_{\mathrm{threshold}}}{3\chi^4 N_p \left(N_p - 1\right)} \right)^{1/4} \sigma f \sqrt{\frac{\delta t}{T}}$$

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Monochromatic Source

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0

Distance

Monochromatic Source

Frequentist approach (time domain):

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0 0

Spin

Monochromatic Source

Frequentist approach (time domain):

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Tends to a power law in the high frequency limit

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0 0 0

Spin-down

Monochromatic Source

Frequentist approach (time domain):

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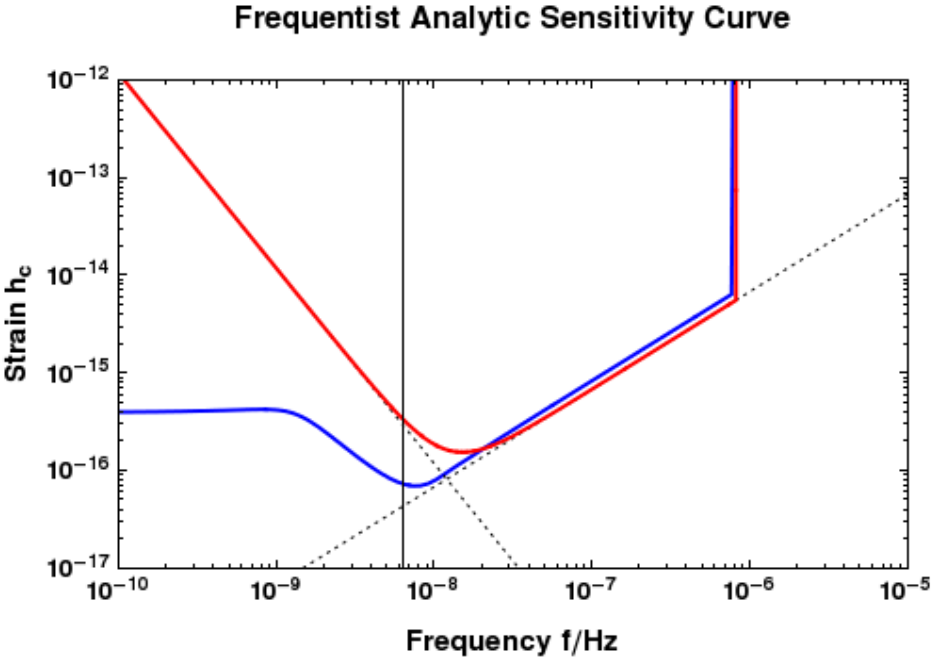
0 0 0

Also a power law in the low frequency limit

$$h_c^{\mathrm{LOW}}(f) \approx \left(\frac{1053\varrho^2_{\mathrm{threshold}}}{128\pi^{12}\chi^4 N_p \left(N_p - 1\right)}\right)^{1/4} \sigma f^{-2} \sqrt{\frac{\delta t}{T}} T^{-3} \sec(\phi)$$

Monochromatic Source

Frequentist approach (time domain):



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Bayesian approach:

$$
\begin{aligned}
s_x(t) &= m_x(\vec{\Theta}'_x, t) + h_x(\vec{\Psi}', t) + n_x(t) && \text{where } x = 1, 2, \ldots, N_p, \\
h_x(\vec{\Psi}, t) &= \frac{\chi h_c}{f} \sin\left(2\pi f t + \phi\right) && \text{with source parameters } \vec{\Psi}^{\mathrm{T}} = \{h_c, f, \phi\} \\
m_x(\vec{\Theta}_x, t) &= \vec{\Theta}_x^{\mathrm{T}} \cdot \vec{N} && \text{with } \vec{N}^{\mathrm{T}} = \left\{1, t, t^2\right\} \text{ and } \vec{\Theta}_x^{\mathrm{T}} = \{\alpha_x, \beta_x, \gamma_x\}
\end{aligned}
$$

Monochromatic Source

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$$\log \mathcal{L}_n(\vec{\Theta}_x) = \log A - \frac{1}{2} \sum_x \left(s_x - m_x(\vec{\Theta}_x) | s_x - m_x(\vec{\Theta}_x) \right) \quad \text{where } (a|b) = \int_0^T \mathrm{d}t \, \frac{a(t)b(t)}{S_n^x}$$

$$\log \mathcal{L}_h(\vec{\Theta}_x, \vec{\Psi}) = \log A - \frac{1}{2} \sum_x \left(s_x - h_x(\vec{\Psi}) - m_x(\vec{\Theta}_x) | s_x - h_x(\vec{\Psi}) - m_x(\vec{\Theta}_x) \right)$$

$$\mathcal{B} \equiv \frac{\mathcal{O}_h}{\mathcal{O}_n} \qquad \overline{\mathcal{B}} = \int \mathrm{d}\mathbf{n} \, P(\mathbf{n}) \mathcal{B} = \exp\left(\sum_x \left[(h'_x | h'_x) - \left(\vec{N}_x^{\mathrm{T}} | h'_x \right) \overleftrightarrow{N}_x^{-1} \left(\vec{N}_x | h'_x \right) \right] \right)$$

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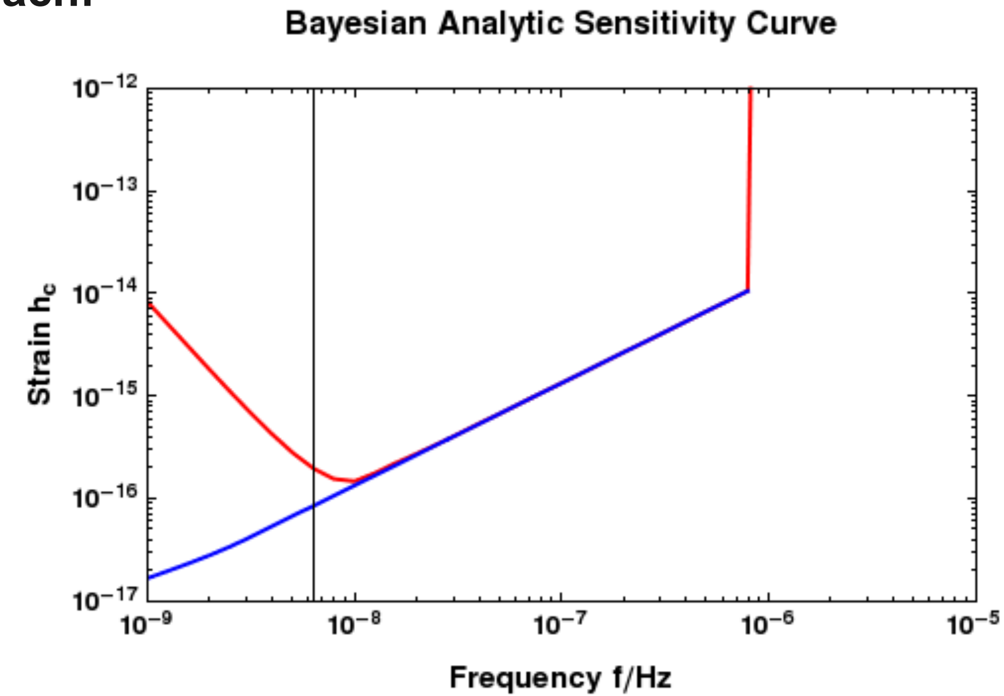
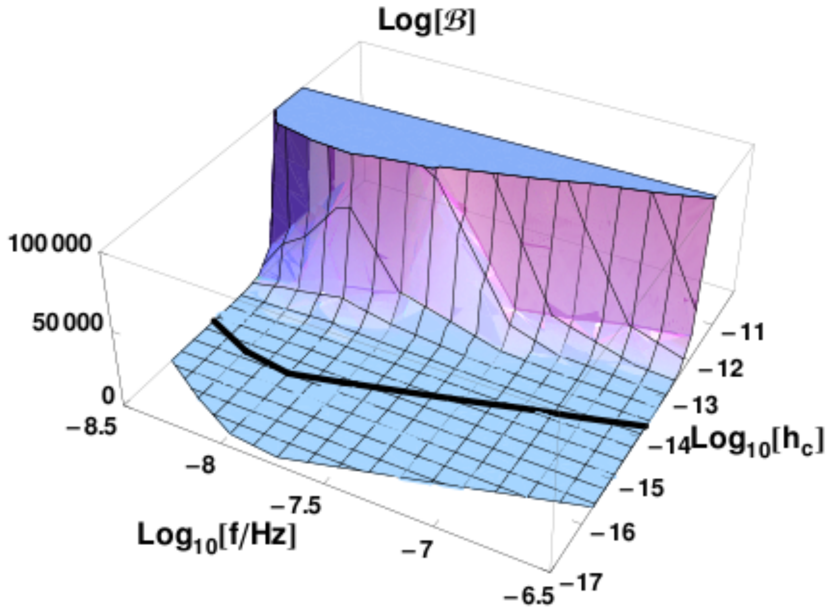
Bayesian approach:



Monochromatic Source

Numerical Bayesian Approach:

- Use the PALSIMULATION code to inject a signal into the TOAs
- Repeat the injections for a range of distances and frequencies
- Plot the Bayes' factor as a surface
- The contour $\mathcal{B} = \mathcal{B}_{\rm th}$ gives the sensitivity curve



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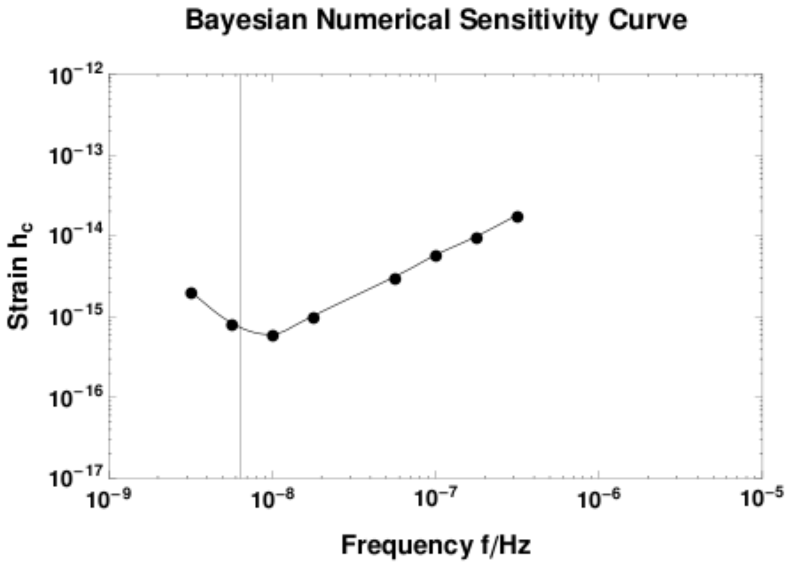

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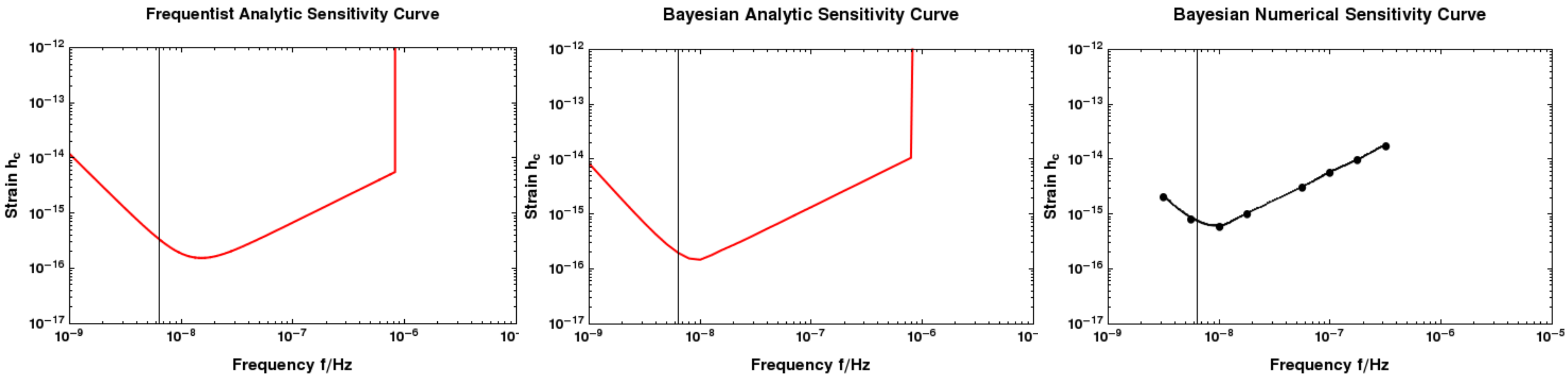

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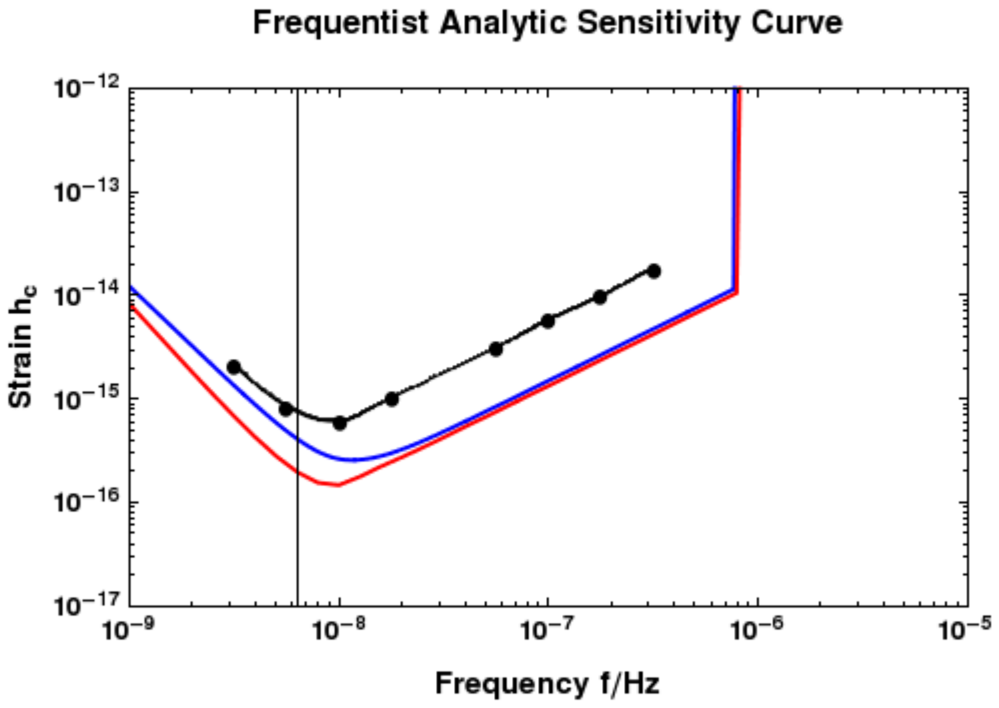

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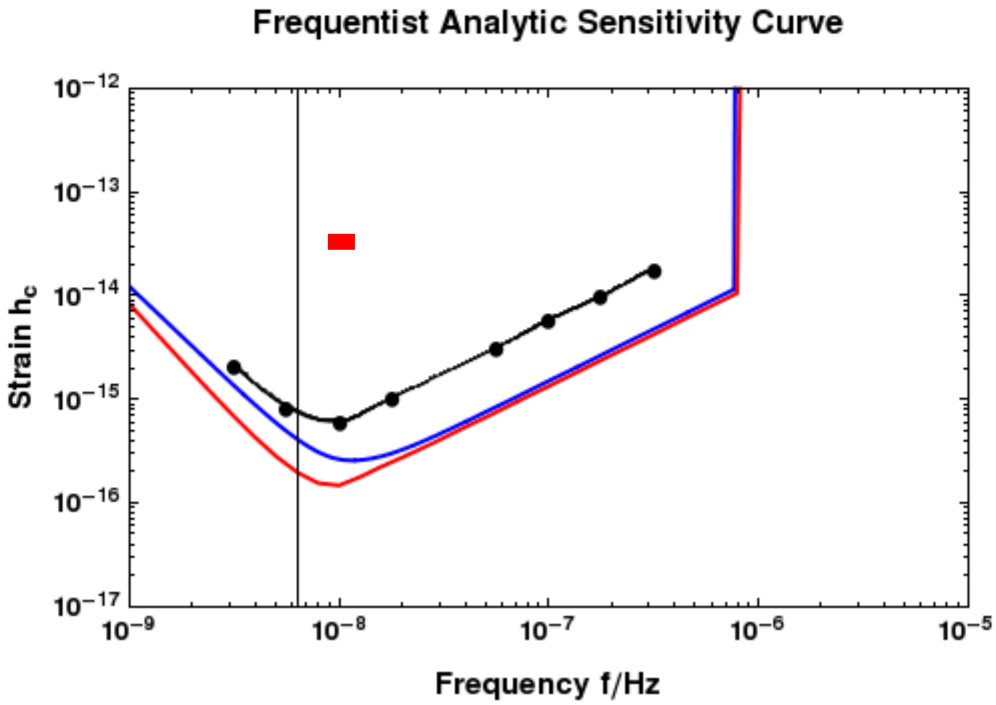

Stochastic Background

Cannot produce a matched filter, only have knowledge of the signal PSD

$$\left\langle \tilde{h}_x(f)\tilde{h}_y^*(f')\right\rangle = \frac{1}{2}\delta(f-f')\Gamma_{xy}S_h(f) \qquad \text{where} \qquad S_h(f) = \frac{A^2}{f_0}\left(\frac{f}{f_0}\right)^{2\alpha-1}$$

Signal contains all frequencies, so determine amplitude, A , in terms of slope, α , then use $h_c = A\left(f/f_0\right)^{\alpha}$ to plot power law integrated sensitivity curves.

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These are the power-law-integrated-sensitivity curves of Thrane and Romano

Thrane, E., & Romano, J. D. 2013, Phys. Rev. D, 88, 124032

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Can repeat the analytic calculations from above to give a similar set of sensitivity curves

Thrane, E., & Romano, J. D. 2013, Phys. Rev. D, 88, 124032

Stochastic Background

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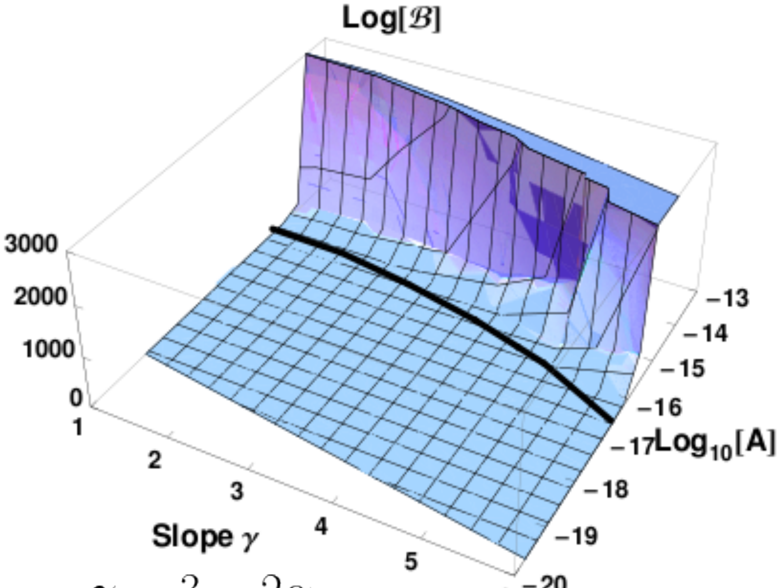
- For each $\{A, \gamma\}$ on this line can plot the line $h_c = A\,(f/f_0)^{\alpha}$



$\gamma = 3 - 2\alpha$

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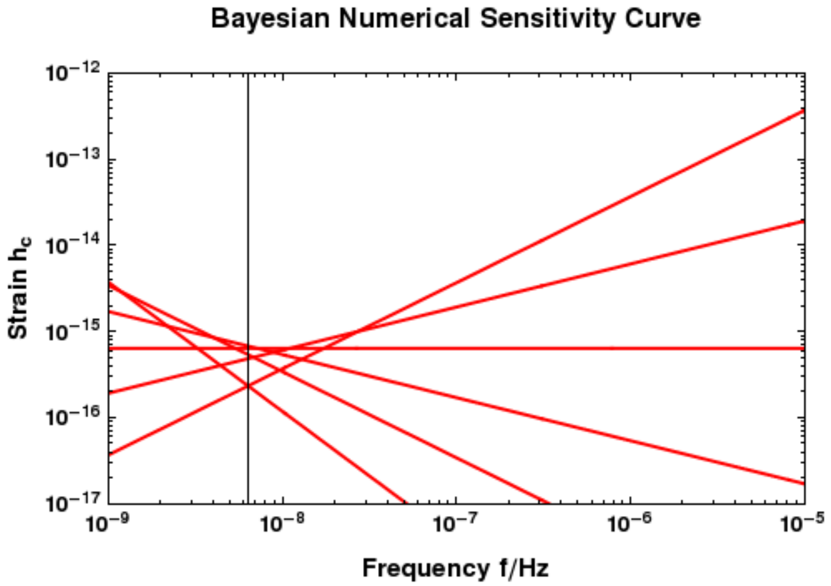

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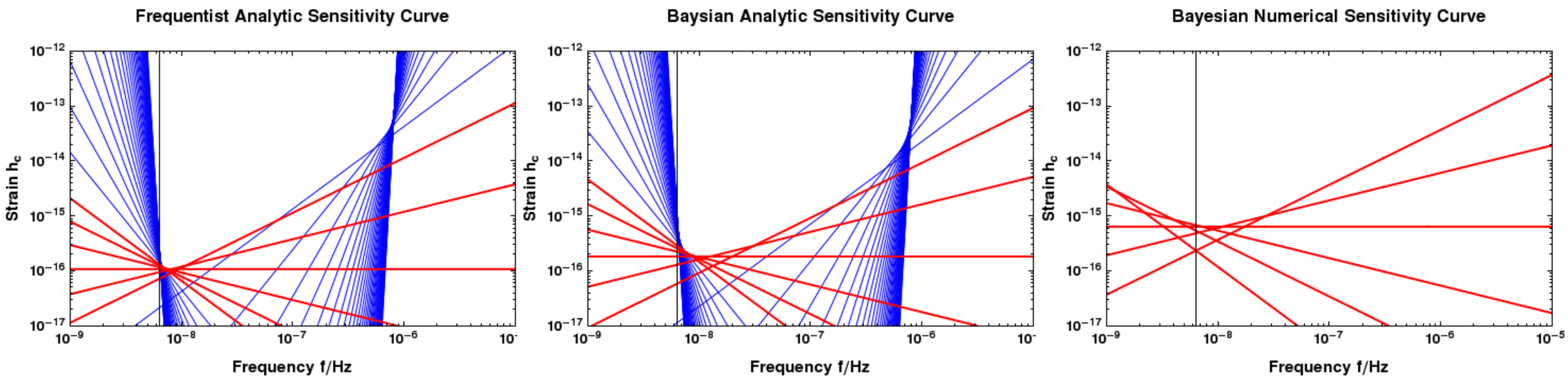
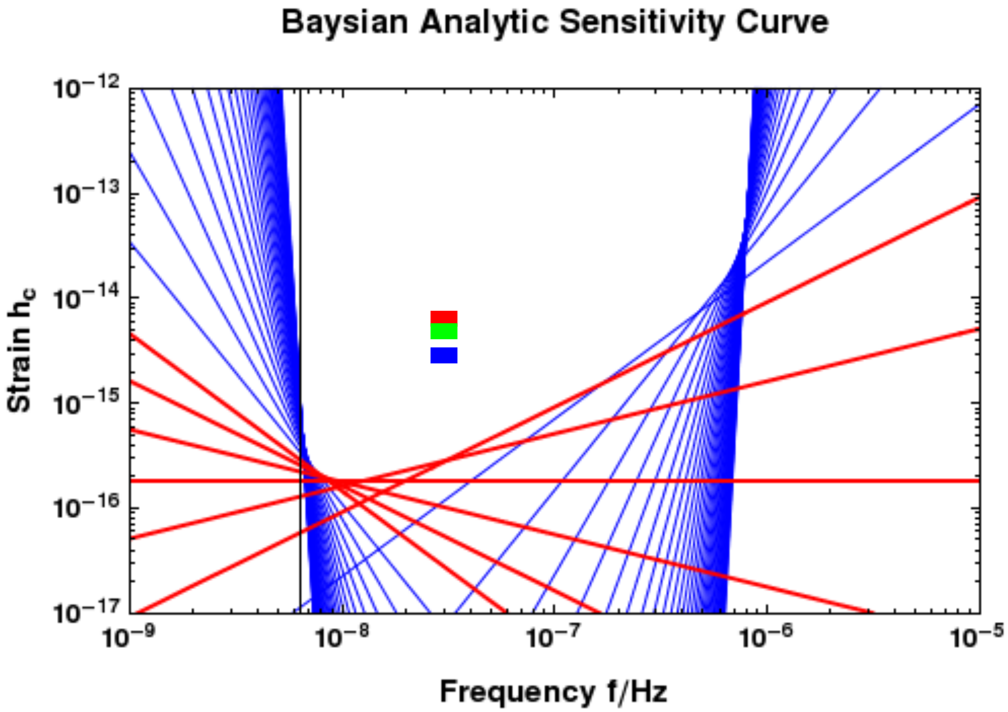


Stochastic Background

NANOGrav

EPTA

PPTA



Conclusions

- The sensitivity curve depends on the properties of the source and the detector
- The shape of the sensitivity curve can be understood using simple analytic arguments in either a Bayesian or frequentist approach
- The different approaches were found to be in good agreement and also in agreement with the numerical calculations