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Anyons: Field Theory and Applications

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Abstract

This paper presents the concept of anyons and how they arise in $d = 2 + 1$ physical theories. Classical properties are briefly discussed, but the main applications are a consequence of the quantum nature of these strange particles. Anyons are particles that occur in planar physics which carry neither integer (bosons) or half integer (fermions) spin. In fact, they can carry any positive real number spin and this paper shows how this manifests. The main physical application addressed is the fractional quantum hall effect (FQHE). Problems with the non-relativisic formulations of anyons is motivation to come up with a relativistic field theoretic way of treating anyons.

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1 Introduction

This paper is intended to convey the peculiarities of physical systems constrained to two space dimensions and explain how these properties come up in physical applications. The motivation for this comes from the applications to the integer quantum hall effect (IQHE) and, more specifically for this paper, the fractional quantum hall effect (FQHE). The latter effect is less understood in terms of solid state techniques, and one may look to the quantum relativist's tool of field theory to understand such an effect.

In $2+1$ dimensions, strange particle states called anyons are present which can, and will, be seen from a straightforward symmetry argument. These particles, unlike bosons or fermions, can have any real value for spin and lead to the idea of generalized statistics in quantum mechanics. Anyons prove to be very useful in describing phenomena like the FQHE.

Anyons can be described in a variety of ways. One of the ways is to look at anyons from a model independent framework and consider them as spinning particles with fractional spin (note here, and for the remainder of the paper, we will stick to the awkward convention of fractional depicting the fact that a value may take on any real number). This can be done in a completely classical setting with no reference to statistics. One may also look at anyons from a field theoretic standpoint as quantum excitations of a classically bosonic or fermionic system. We will primarily be focused on the latter description which will restrict our attention to the celebrated Chern-Simons theories in 2+1 dimensions. Widely known for many interesting topological properties, Chern-Simons is important in this setting as it is the covariant generalization of a particle with generalized statistics and can admit anyon-like excitations at the quantum level.

We will look at Chern-Simons theory from various perspectives and offer few different methods of quantization, each of which have some advantages and disadvantages. Lastly, we will look into the somewhat ill-understood realm of fractional supersymmetry and how this could possibly be used to construct a relativistic field theory of anyons. No new

results are given in this paper, but the methods highlighted above are a relatively new way of solving this problem and will be the topic of future research.

1.1 Conventions

We will primarily be working in $2+1$ dimensions with the choice of metric signature $g_{\mu\nu} = \text{diag}(+1, -1, -1)$. Repeated indices are always summed over unless otherwise stated. The standard three vector is labeled with a Greek index: $x^\mu = (x^0, x^1, x^2) = (t, \mathbf{x})$. Latin indices x^i go over $i = 1, 2$ and are the spatial parts of the vector. Shorthand for derivatives is $\partial_\mu = \frac{\partial}{\partial x^\mu}$.

2 Symmetries in the plane and classical anyons

Here we analyze the symmetries present in $2+1$ dimensions. We start with the Poincaré group and see how its representation leads to the concept of arbitrary spin in the plane. Next, we take a digression to a non-relativstic interpretation of anyons and how it relates to the relativistic notion. This section is devoted to a classical description of anyons which will lead into the quantum concept of anyonic behavior.

2.1 The Poincaré Group

The set of space-time symmetries of quantities in arbitrary dimension are formally defined to be the set of all real linear transformations

$$(a^\mu, \Lambda^\mu_\nu) : x^\mu \mapsto x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \tag{1}$$

which leave $s^2 \equiv g_{\mu\nu} x^\mu x^\nu = (x^0)^2 - (x^1)^2 - (x^2)^2$ invariant. They actually form a group called the Poincaré group. The above transformation law indicates that π is actually a semidirect product of a translation group $N \cong \mathbb{R}^3$ and the three-dimensional Lorentz group $SO(2,1)$. As with any Lie group, we refer to the infinitesimal generators to make

the connection to the associated Lie algebra. The Hermitian generators of the Lorentz subgroup are $M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + S_{\mu\nu}$, and they obey the usual $SO(d,1)$ Lie algebra,

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(g_{\mu\sigma} M_{\nu\rho} - g_{\nu\sigma} M_{\mu\rho} - g_{\mu\rho} M_{\nu\sigma} + g_{\nu\rho} M_{\mu\sigma}). \tag{2}$$

As we shall see in a moment, the spin part $S_{\mu\nu}$ is the defining factor for the uniqueness of $2+1$ dimensional theories. The generators of the translational symmetries are $P_\mu = -i\partial_\mu$, which of course commute with each other, yet have non-trivial bracket with the rotation generators. We can summarize the commutation relations by introducing the (non-Hermitian) generators $K^0 = -iM_{01}$, $K^1 = -iM_{02}$, and $K^2 = M_{12}$. The brackets then read,

$$[K^\mu, K^\nu] = i\epsilon^{\mu\nu\rho} K_\rho \ , \quad [K^\mu, P^\nu] = i\epsilon^{\mu\nu\rho} P_\rho \ , \quad [P_\mu, P_\nu] = 0 \tag{3}$$

which summarize the Poincaré algebra in $2+1$ dimensions. As with the $3+1$ dimensional case, we have two Casimir operators, yet they take a slightly different form. The first is $P^2 \equiv P^\mu P_\mu$, and the second is the Pauli-Lubanski scalar

$$W \equiv P \cdot K = \epsilon^{\mu\nu\rho} P_\mu M_{\nu\rho}. \tag{4}$$

For single particle representations Φ we have the eigenvalue equations

$$P^2 \Phi = m^2 \Phi \ \text{ and } \ W\Phi = -ms\Phi \tag{5}$$

In contrast to the $3+1$ case, there is no *a priori* restriction to the values that the spin s can take on. This conclusion can also be seen from a more formal covering group argument and this is presented in Appendix A[1].

2.2 The Galilei Group and Anyons

We now take a slight detour to demonstrate how the concept of anyons from the classical perspective can lead to unusual implications. The Galilei group $G^{2|1}$ in three dimensions is the group of symmetries which simultaneously leave $\tilde{g}_{\mu\nu}$ =diag$(1,0,0)$ and

$\tilde{g}^{\mu\nu}$ =diag$(0,1,1)$ invariant. We have six generators: J, K_i, H, P_i of rotations, boosts, time translations, and spacial translations, respectively. They satisfy the Poisson brackets (only non-vanishing commutators listed),

$$\{J, K_i\} = \epsilon_{ij} K_j \ , \quad \{J, P_i\} = \epsilon_{ij} P_j \ , \quad \{K_i, H\} = P_i. \tag{6}$$

It is known that $G^{2|1}$ admits a two-fold central extension via the modifications,

$$\{K_i, P_j\} = m\delta_{ij} \quad \text{and} \quad \{K_i, K_j\} = -\kappa\epsilon_{ij}. \tag{7}$$

We have two Casimir invariants,

$$C_1 = mJ + \kappa H - \epsilon_{ij} K_i P_j \ \text{ and } \quad C_2 = mH - \frac{1}{2} P_i P_i. \tag{8}$$

For the case $C_1 = C_2 = 0$ we have the following realization of the algebra, $P_i = p_i$, $K_i = mx_i - tp_i + m\theta\epsilon_{ij} p_j$, $J = \epsilon_{ij} x_i p_j + \frac{1}{2}\theta p_i p_i$, and $H = \frac{1}{2m} p_i p_i$ where $\theta \equiv \kappa/m^2$. With this, we have the usual time evolvement $\dot{x}_i = \{x_i, H\} = \frac{p_i}{m}$, yet coordinates do not commute [2]:

$$\{x_i, x_j\} = \theta\epsilon_{ij}. \tag{9}$$

We can get to this same structure among coordinates by starting with Poincaré symmetry and by placing no restrictions on the spin. We obtain the result after demanding covariance,

$$\{x_\mu, x_\nu\} = s\epsilon_{\mu\nu\rho} \frac{p^\rho}{(p^2)^{3/2}} \tag{10}$$

[3]. We can make the non-relativistic expansion $p_0^2 = mc + \frac{p_i p_i}{2mc} + \mathcal{O}\left(\frac{1}{c^2}\right)$ to get

$$\{x_\mu, x_\nu\} \approx \frac{s\epsilon_{ij}}{m^2} \tag{11}$$

in the non-relativistic limit [4]. We conclude there is a relationship between these two situations with the correspondence $\kappa \leftrightarrow s$.

3 Quantum Anyons

We now look at anyons as particle states that have fractional spin in a statistical sense which is a consequence of their quantum nature.

3.1 Fractional spin through quantum solenoids

In this example, we show how the spectrum of angular momentum can be shifted from integer or half integer multiples of the fundamental constants [5]. This is not an actual realization of fractional spin, but with some modification we can exhibit such an effect. Consider the planar motion of a charged particle and let there be a solenoid centered at the origin. The Lagrangian is,

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 + e\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}) \tag{12}$$

where we can gauge the potential to be

$$A^i(x) = -\frac{\Phi}{2\pi}\epsilon^{ij}\frac{x^j}{\mathbf{x}^2} \tag{13}$$

[6]. The magnetic field is the scalar quantity given by $B = \epsilon^{ij}\partial_i A_j = \Phi\delta^{(2)}$; therefore, the quantity Φ can just be interpreted as the magnetic flux. The momenta are found to be $p^i = m \cdot x^i + eA^i$ and the Hamiltonian has the form

$$H = \frac{1}{2m}\left(p^i - eA^i\right)^2 = \frac{1}{2}m\dot{\mathbf{x}}^2 \tag{14}$$

which, as expected, is the same as that of a free particle. The presence of the singular solenoid at the origin does not affect the classical dynamics of the particle, but it does give rise to quantum effects. Note that, because the potential outside the solenoid is constant we can set it to zero via a gauge transformation $A_i' = A_i - \partial_i\Omega$ where $\Omega = \frac{\phi}{2\pi}\Phi$ and ϕ is the polar angle. The wave function becomes $\psi' = e^{-i\Omega}\psi$. If we demand ψ to satisfy periodic boundary conditions: $\psi(\phi) = \psi(\phi + 2\pi)$, then ψ' satisfies "twisted" boundary conditions: $\psi(\phi + 2\pi) = e^{-i\Omega}\psi(\phi)$. The mechanical angular momentum operator is the usual $J_m = -i\partial_\phi$ and we see that

$$J_m\psi' = \left(l - \frac{\Phi}{2\pi}\right)\psi' \tag{15}$$

where $J_m\psi = l\psi$ is the usual spectrum. Thus, we see that in the presence of these fictitious solenoids our wave function acquires some non-trivial shift in the spectrum of angular momentum; that is, fractional spin.

We can, in a sense, make this construction more formidable by not biasing our shift in the angular momentum spectrum to rotations about the origin [7]. For this we introduce a many particle generalization of the above construction. The first thing to notice is that we can write the interaction term in the above Lagrangian as $L_{int} = \frac{\Phi}{2\pi}\dot{\Theta}(x)$ where

$$\Theta(x) = \arctan\left(\frac{x^2}{x_1}\right). \tag{16}$$

This function is obviously multi-valued on $\mathbb{R} \setminus \{0\}$ but by defining a branch cut we can make it single valued on the universal cover $\widetilde{\mathbb{R}^2 \setminus \{0\}}$. We can write the total Lagrangian for N particles as

$$L = \sum_{i=1}^{N} \frac{1}{2} m \dot{\mathbf{x}}_i^2 + \frac{\Phi}{2\pi} \sum_{i \neq j} \dot{\Theta}_{ij} \tag{17}$$

where we have used the notation $\Theta_{ij} \equiv \Theta(\mathbf{x}_i - \mathbf{x}_j)$. The spectrum is now $J = l - \frac{\Phi}{2\pi} N(N-1)$ for $l \in \mathbb{Z}$. We can also look at this system in the Hamiltonian formalism and we find that the system (H, ψ) where H is the Hamiltonian associated with the above Lagrangian is equivalent to the system (through a gauge transformation) (H_{free}, ψ') where H_{free} is the free Hamiltonian and

$$\psi'(\mathbf{x}_i) = \prod_{i<j} \exp\left[-i\frac{\Phi}{\pi}\Theta_{ij}\right] \psi(\mathbf{x}). \tag{18}$$

3.2 The covariant formulation

Some problems exist with the previous formulation. First, the interaction lagrangian, as written, is nonlocal. Secondly, the gauge field $\mathbf{A}$ above is not a genuine gauge field because it is not even dynamical. We fix this problem, and by doing so we will run into the notorious Chern-Simons term. For this, we consider coupling n charged particles to an external gauge field A_μ [7]. In some non-relativistic limit we see that the same type of generalized statistics is achieved. We start with an n particle source, j_μ. We couple to a gauge field via $L = L_0 + L_I + L_g$ where,

$$L_I = e\sum_i (\dot{\mathbf{x}}_i \cdot \mathbf{A} - A^0) \text{ and } \quad L_g = \frac{\pi}{\Phi} \int d^2\mathbf{y} \left[\mathbf{A}(\mathbf{y}) \times \dot{\mathbf{A}}(\mathbf{y}) + A^0(\mathbf{y}) B(\mathbf{y})\right]. \tag{19}$$

We can write the above in a covariant fashion,

$$S_I = \int d^3x\ j^\mu(x) A_\mu(x) \text{ and } \quad S_g = \frac{\pi}{\Phi} \int d^3x\ \epsilon^{\mu\nu\rho} A_\mu(x) \partial_\nu A_\rho(x). \tag{20}$$

We have finally seen the Chern-Simons term: $\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$. Just as we saw peculiarities group theoretically in $2+1$ dimensions, we have this term which is only possible in odd dimensions and not in the usual 3+1 dimensional setting. This term has expanded to create and entire branch of theoretical physics with its importance in topological field theories, knot theory, and more. It can be seen most easily as the surface term of the Lorentz invariant term in $3+1$ dimensions $\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$:

$$\int d^4x\ \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = 4 \oint d^3x\ \epsilon^{0\mu\nu\rho} A_\mu \partial_\nu A_\rho \tag{21}$$

which, in most circumstances, can be set to vanish. When the four-volume is chosen to be compact or finite, this term can affect physics on a $2+1$ dimensional surface.

It can be shown through the path integral method that this system acquires fractional statistics. The non-relativistic effective action takes the form

$$S_{eff} = \frac{\Phi}{2\pi} \sum_{i,j} \int dt\ \frac{d}{dt} \Theta_{ij} + S_g \tag{22}$$

where S_g is a topological term which does not affect the spin or statistics of the system [7].

4 Anyons and field theory

In this section, we will discuss the need for a field theoretic formulation of anyons. To do so we will briefly discuss the quantum hall effect and the solutions of variations of the effect and how completeness is not achieved with the standard methods of trial wave-functions and effective field theories.

4.1 The FQHE

The classical Hall effect is a simple consequence of the Lorentz force law. Consider a constant magnetic field in the $\hat{\mathbf{z}}$ direction $\mathbf{B} = B\ \hat{\mathbf{z}}$ and an electron confined to a plane

perpendicular to this field. The Lorentz force is just, $\mathbf{F} = \frac{eB}{c}(v_x\hat{\mathbf{y}} - v_y\hat{\mathbf{x}})$. We suppose that a batch of electrons have current $\mathbf{j} = \rho e v_x \hat{\mathbf{x}}$ where $\rho = \frac{N}{A}$, N is the number of electrons and A is the area of the plane. There is an induced electric field $\mathbf{E} = \frac{1}{c}v_x B\hat{\mathbf{y}}$ and we can define the so-called Hall conductance as,

$$\sigma_H \equiv \sigma_{xy} = \frac{ec}{B}\rho. \tag{23}$$

An important fact to note is that this Hall conductance varies continuously with respect to the magnetic field B. The quantum Hall effect is prevalent in a high magnetic field (or low temperatures) and predicts the Hall conductance to acquire plateaus. That is, the Hall conductance is given by

$$\sigma_H = \frac{2\pi\nu e^2}{\hbar} \tag{24}$$

where ν is a rational number. When ν is an integer, this is known as the integer quantum Hall effect (IQHE), and when ν is not an integer, it is called the fractional quantum Hall effect (FQHE).

4.2 The Laughlin wave-function

The IQHE was observed in 1980, and two years later the FQHE was observed [8]. The IQHE is easily described using a Landau level approach and quantizing this way. The FQHE, however, is believed to be a many-body effect and can not be described through such techniques. Laughlin found a variation wavefunction for the ground state of the FQHE which worked nicely in predicting the fractional levels of the conductivity. Specifically, the ground state Laughlin wave-function for N particles takes the form

$$\psi(z_1, \ldots, z_N) = \prod_{j<k}(z_j - z_k)^{2q+1} \exp\left\{-\sum_i \frac{|z_i|^2}{4\ell_m^2}\right\} \tag{25}$$

where $z = x + iy$, $q \in \mathbb{Z}_+$, and $\ell_m^2 = \hbar/eB$ is the so called magnetic length [8]. One can compute the Hall conductance by perturbing the corresponding Hamiltonian adiabatically with respect to the time dependence of the magnetic flux.

It was confirmed by Wilczeck that the ground state actually acquires anyonic statistics by computing the Berry phase of the exchange of two quasi-holes, as to be compared with the fictitious solenoids discussed earlier. Incompleteness of the Laughlin approach arises when one tries to analyze the full symmetry of the QH system. This problem has been addressed and a field theoretic approach to anyons is believed to hold the solution.

4.3 Anyons as fundamental fields

In both the relativistic and non-relativistic case, work has been done to produce soliton solutions to Chern-Simons models which acquire anyonic spin and statistics. This is unsatisfactory if one wants a local theory for anyons and it is not the approach that we discuss here. We review over the tools needed to develop anyons as fundamental objects in a completely local quantum field theory in $2+1$ dimensions. Knowledge of the symmetry group in $2+1$ dimensions was discussed earlier and plays a very important factor in this formulation.

It is constructive to first see how a formalization looks in a non-relativistic theory and where the problems arise. Let φ be a complex bosonic field and take the Lagrangian

$$\mathcal{L} = i\varphi^* D_0\varphi + \frac{1}{2m} D_i\varphi^* D_i\varphi + \frac{\kappa}{2}\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \tag{26}$$

where $D_\mu = \partial_\mu - iA_\mu$ is the covariant derivative [8]. Varying with respect to A_0 we get the Gauss' law constraint: $B = -\frac{e}{\kappa}\rho$ where $\rho \equiv \phi^*\phi$ is the number-density and $B = \nabla \times A = \epsilon_{ij}\partial_i A_j$. We choose the transverse gauge $\partial_i A^i = 0$ to solve the Gauss' law constraint:

$$\mathbf{A} = \frac{e}{\kappa}\frac{\nabla}{\nabla^2} \times \rho = \frac{e}{\kappa}\epsilon^{ij}\frac{\partial_j}{\nabla^2}\rho \tag{27}$$

or more concisely,

$$A^i(\mathbf{x}) = \epsilon^{ij}\partial_j \left(\frac{e}{\kappa} \int d^2y \; G(\mathbf{x}-\mathbf{y})\rho(\mathbf{y}) \right) \tag{28}$$

where $\nabla^2 G(\mathbf{x}-\mathbf{y}) = \delta^{(2)}(\mathbf{x}-\mathbf{y})$. We can write this in terms of the well known angle function $\Theta(\mathbf{x})$,

$$\mathbf{A}(\mathbf{x}) = -\frac{e}{2\pi\kappa} \int d^2y \; \nabla_x \Theta(\mathbf{x}-\mathbf{y})\rho(\mathbf{y}). \tag{29}$$

As discussed earlier, the Θ function is multi-valued and we cannot, in general, take out the gradient from the integral in order to make A^i a pure gauge field. To fix multi-valuedness, we can specify some branch cut in the (y^1, y^2) plane starting at (x^1, x^2), yet we still cannot remove the derivatives, for the branch cut will also depend on x. The special case where we can write A^i as a total gradient is when the number-density is a sum of δ-functions, i.e. the non-relativistic case. In this case we have

$$\mathbf{A}(\mathbf{x}) = -\frac{e}{2\pi\kappa}\nabla_x \left(\int d^2y \; \Theta_{xy} \, \rho(\mathbf{y}) \right). \tag{30}$$

Similarly, we can show that the scalar potential can also be written as a pure gauge and we find that we can make a gauge transformation $A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda$ to eliminate the gauge field from our classical theory. Under this gauge transformation the action takes the form

$$S' = \int d^3x \; \left[i\tilde{\varphi}^* \partial_0 \tilde{\varphi} + \frac{1}{2m} \tilde{\varphi}^* \partial_i \partial_i \tilde{\varphi} \right] \tag{31}$$

where the gauge transformed matter fields are $\tilde{\varphi} = \exp[-ie\Lambda(x)]\varphi(x)$. With the usual bosonic equal time commutation relations

$$[\varphi(\mathbf{x}), \varphi^\dagger(\mathbf{y})] = \delta^{(2)}(\mathbf{x} - \mathbf{y}) \tag{32}$$

$$[\varphi(\mathbf{x}), \varphi(\mathbf{y})] = 0 = [\varphi^\dagger(\mathbf{x}), \varphi^\dagger(\mathbf{y})] \tag{33}$$

the gauge adjusted matter field commutation relations are

$$\tilde{\varphi}(\mathbf{x})\tilde{\varphi}(\mathbf{y}) = e^{i\pi\alpha} \tilde{\varphi}(\mathbf{y})\tilde{\varphi}(\mathbf{x}) \tag{34}$$

$$\tilde{\varphi}^\dagger(\mathbf{x})\tilde{\varphi}(\mathbf{y}) = e^{i\pi\alpha} \tilde{\varphi}(\mathbf{y})\tilde{\varphi}^\dagger(\mathbf{x}) + \delta^{(2)}(\mathbf{x} - \mathbf{y}) \tag{35}$$

where, implicitly we have chosen a cut so that $\Theta(\mathbf{e}_1 + \epsilon\mathbf{e}_2) \to 0$ and $\Theta(\mathbf{e}_1 - \epsilon\mathbf{e}_2) \to 2\pi$ as $\epsilon \to 0$. Note this sets $\Theta_{xy} - \Theta_{yx} = \pi$. Also, we have set $\alpha = e^2/2\pi\kappa$. The above shows that these bosonic fields acquire anyonic statistic under this gauge transformation. This second quantized theory can be shown to be equivalent to the first quantized model discussed earlier where we considered A_μ as a statistical interaction between particles in

the plane. The problem with this construction is that by demanding these operators to be local we must choose a specific branch cut to come up with the commutation relations. This is simply unsatisfactory for a physical theory, and to encode the way by which the wave-function changes, we must take into account how we braid the interchanging of x and y; this requires non-local operators. A way to do this is to choose a single-valued definition for Θ and hence demote $\tilde{\varphi}$ to a non-local operator. To conclude, for the non-relativistic case, the only way we can field theoretically describe anyons is through a non-local field theory.

So far, no local relativistic quantum field theory for anyons, but unlike the non-relativistic case the possibility must not be ruled out. For the quantum theory, we are interested in finding quantized anyon operators which give rise to states with arbitrary spin. The prototype model for this is the pure Chern-Simons term coupled to a scalar matter field. The Lagrangian is given by,

$$\mathcal{L} = (D_\mu \varphi)^* (D^\mu \varphi) + \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \tag{36}$$

This model is canonically quantized in both equal times and in the light-cone in Appendix B. For now we will be interested in the equal time quantization. It can be shown that this model is Poincaré covariant with the brackets and constraints given in the appendix. The physical states of the model are the states which are annihilated by our first class constraints $\Omega_0|\psi\rangle = \bar{\xi}|\psi\rangle = 0$. We define the one particle states to be the ones which carry unit charge: $q = \int d^2x\ j_0$. That is, the state $|1\rangle = \hat{\varphi}|0\rangle$. The creation operator must then satisfy

$$[j_0(\mathbf{x}), \hat{\varphi}(\mathbf{y})] = \delta^{(2)}(\mathbf{x} - \mathbf{y})\hat{\varphi}. \tag{37}$$

The creation anyon operator $\hat{\varphi}$ actually has the explicit form

$$\hat{\varphi}(\mathbf{x}) = \exp\left\{ \frac{i\pi}{\theta} \int d^2y\ \Theta(\mathbf{x} - \mathbf{y}) j_0(\mathbf{y}) + i \int^x d\mathbf{y} \cdot \mathbf{A}(\mathbf{y}) \right\} \varphi \tag{38}$$

where $\theta = 2\pi^2\kappa$ [9]. Using the branch cut, $\Theta_{xy} - \Theta_{yx} = \pi$ we can invoke the Baker-Campbell-Hausdorff formula $\exp\{\int d^2\xi f(x-\xi)g(\xi)\}\varphi(y)\exp\{-\int d^2\xi f(x-\xi)g(\xi)\} = \exp[f(x-$

$y)]\varphi(y)$ to realize the generalize statistics relation

$$\hat{\varphi}(x)\hat{\varphi}(y) = e^{i\pi^2/\theta}\hat{\varphi}(y)\hat{\varphi}(x). \tag{39}$$

This shows the spin-statistics connection for this theory. Note that $\theta = \pi/2n$ and $\theta = \pi/(2n+1)$ correspond to bosons and fermions respectively while other values of θ, which are not prohibited, correspond to anyons. To see explicitly the non-locality of the theory in terms of anyonic fields we rewrite

$$\begin{aligned}[D_i\varphi(x)]^*[D^i\varphi(x)] &= \partial_i\left[\exp\left\{-\frac{i\pi}{\theta}\int d^2y\ \Theta(x-y)\hat{j}_0(y)\right\}\hat{\varphi}(x)\right]^* \times \\ &\quad \partial_i\left[\exp\left\{-\frac{i\pi}{\theta}\int d^2y\ \Theta(x-y)\hat{j}_0(y)\right\}\hat{\varphi}(x)\right]\end{aligned}$$

[9]. If we were to work in the light-cone formalism, we would see a very similar non-locality property in an anyon Hamiltonian, but it happens that the Hamiltonian has a much simpler form. In future research this might be a desirable property when trying to localize the anyon fields with techniques such as fractional supersymmetry.

5 Remark on Fractional SUSY

Fractional supersymmetry is a relatively new mathematical idea and an even newer physical principle. In ordinary supersymmetry, we extend the Poincaré algebra to include a fermionic generator Q (and its conjugate) which, when acting on the appropriate Hilbert space, replaces a boson field for a fermion field and vice versa. The natural bracket for this fermionic generator is the anticommutator. This gives what is called a $\mathbb{Z}_2$-graded structure to the algebra. Supersymmetry can be seen in a natural way through the superspace formalism where these generators actually arise as symmetries in a new space - superspace - (x^μ, θ) where x^μ is the usual space-time coordinate and θ is the super-coordinate which is taken to be a Grassmann variable. That is, it satisfies $\theta^2 = 0$. Constructions such as derivation and integration on the supermanifold (x^μ, θ) can be done, and we can build any supersymmetric theory through this formalism.

Because we are interested in not only bosons or fermions, but particles of any spin, one may conjecture a more general type of supersymmetry. Instead of a $\mathbb{Z}_2$-graded algebraic structure we are looking for a $\mathbb{Z}_F$-graded structure, and this is known as F-supersymmetry or FSUSY for short [10]. A straightforward generalization of the superspace formalism would give us a space (x^μ, θ) which satisfies $\theta^F = 0$. We might then ask what is the natural bracket for our algebra. One may take various approaches to construct such a fractional superspace and it is very much a work in progress. The goal would be to apply such techniques to anyon field theories.

6 Conclusion

To conclude, we have shown how anyons arise in various 2+1 dimensional physical systems. While the IQHE can be described with ordinary techniques we saw that anyons might hold the answer to a more complete description of the FQHE, and specifically a field theoretic formulation of them. The importance of the Chern-Simons term enters as a consequence of making anyonic interactions covariant. Non-relativistic field theories for anyons as fundamental fields were seen to possess intrinsic non-localities. Relativistically speaking we saw the same type of problem, but as remarked in the end of the last section it is still an open problem to find a localized relativistic theory for anyons.

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Appendix

A The universal cover of $SO(2,1)$

Suppose our quantum system is described by the Hilbert space $\mathcal{H}$ and thus the states are elements of the projective space $P(\mathcal{H}) = (\mathcal{H} \setminus \{0\})/C^\times$. A symmetry of our system is a bijection $P(\mathcal{H}) \to P(\mathcal{H})$ which leaves the quantity $|\langle \psi, \bar{\psi} \rangle|^2$ invariant. The set of all symmetries is denoted $S(\mathcal{H})$. A famous theorem by Wigner states that all symmetries of our quantum system are induced by unitary (or anti-unitary) operators acting on $\mathcal{H}$. Moreover these symmetries $S(\mathcal{H})$ form a group. If G is a Lie group, it is known that the connected component of G is generated by elements of the form $\exp X$ where X is an element of the Lie algebra of G. Also, all elements of the connected component of G act as unitary symmetries. We have the following result from Wigner

Theorem A.1 *If G is a connected Lie group and $\lambda : G \to S(\mathcal{H})$ is a group homomorphism then for each $g \in G$ there is a unitary operator $L(g)$ such that $\lambda(g)$ is induced by $L(g)$.*

In this manner we obtain a representation of G in $\mathcal{H}$: $L : G \to U(\mathcal{H})$ where $U(\mathcal{H})$ is the unitary group of symmetries. A *projective unitary representation* of G in $\mathcal{H}$ is a continuous homomorphism $\tilde{L} : G \to U(\mathcal{H})/\mathbb{C}^\times$.

A given Lie algebra $\mathfrak{g}$ uniquely describes a simply connected Lie Group $\widetilde{G}$ (all paths in $\widetilde{G}$ are contractible). Moreover, any Lie group can be obtained from a simply connected Lie group via the quotient of some discrete central subgroup $N \lhd G$: $G = \widetilde{G}/N$. This simply connected group $\widetilde{G}$ is called the *universal cover* of G. An irreducible unitary representation of the covering group uniquely defines a projective unitary representation of G. It therefore suffices to find unitary representations of the universal cover of any given Lie group. For rotations in three dimensions we have $\widetilde{SO(3)} = SU(2)$, for the ordinary Lorentz group in four dimensions, $\widetilde{SO(3,1)^+} = SL(2,\mathbb{C})$. Take for instance the little group corresponding to

$P_\mu P^\mu = m^2 > 0$ of the three dimensional Poincaré group $\pi^+ = \mathbb{R}^3 \ltimes SO(3,1)^+$. That is,

$$\{p_\mu \ : \ p^2 = (p^0)^2 - (p^1)^2 - (p^2)^2 = m^2 > 0\}. \tag{40}$$

This group is isomorphic to $O(2)$ the orthogonal 2×2 matrices. A group element has the form

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{41}$$

Obviously, there is a multi-valuedness in the corresponding representation and thus the little group is not simply connected. In fact, $\widetilde{O(2)} = \mathbb{R}$. Which most formally depicts the arbitrariness of the eigenvalue of the spin operator in $2+1$ dimensions to be any real number. Intuitively, this is obvious as the group of rotations in $\mathbb{R}^d$ is abelian for $d \leq 2$.

B Pure Chern-Simons coupled to a scalar

Consider the theory of a Chern-Simons term with a scalar field,

$$\mathcal{L} = (D_\mu \varphi)^* (D^\mu \varphi) + \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \tag{42}$$

where we have the covariant derivative $D_\mu \phi \equiv (\partial_\mu + ieA_\mu)\varphi$. We will quantize this model canonically using the Dirac procedure [9] for constrained systems in equal time and in the light-cone. There are advantages and disadvantages in both methods.

B.1 Equal-Time Qauntization

The conjugate momenta are,

$$\begin{aligned} \pi &= \frac{\partial L}{\partial \dot{\varphi}} = (D_0 \varphi)^* \ , \qquad \pi^* = D_0 \varphi \\ \pi^0 &= \frac{\partial L}{\partial \dot{A}_0} = 0 \ , \qquad \pi^i = \frac{\partial L}{\partial \dot{A}_0} = -\kappa \epsilon^{ij} A_j. \end{aligned} \tag{43}$$

The conserved gauge covariant current is, $j_\mu = ie(\varphi^* D_\mu \varphi - \varphi D_\mu^* \varphi^*)$. We postulate the equal-time Poisson brackets,

$$\{\phi(\mathbf{x}), \pi(\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y}) \ , \qquad \{\phi^*(\mathbf{x}), \pi^*(\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y}) \tag{44}$$

$$\{A_\mu(\mathbf{x}), \pi_\nu(\mathbf{y})\} = -g_{\mu\nu}\delta(\mathbf{x}-\mathbf{y}). \tag{45}$$

These lead to the primary constraints,

$$\Omega_0 \equiv \pi_0 \approx 0 \qquad \text{and} \qquad \Omega_i \equiv \pi^i + \kappa\epsilon^{ij}A_j \approx 0. \tag{46}$$

The quantization technique used is known as Dirac quantization and follows most closely the method given in [11].The canonical Hamiltonian density takes the form,

$$\begin{aligned} \mathcal{H}_c &= \sum \pi^a \psi^a - \mathcal{L} && (47)\\ &= \pi\dot{\varphi} + \pi^*\dot{\varphi}^* + \pi^i \dot{A}_i - \mathcal{L} && (48)\\ &= \pi^*\pi + (D^i\varphi)^*(D_i\varphi) + A_0 j_0 - \kappa\epsilon^{ij}(A_0\partial_i A_j + A_i\partial_j A_0). && (49) \end{aligned}$$

As in any constrained system, the canonical Hamiltonian is not the correct one and we must introduce the modified Hamiltonian, $H = \int d^2x(\mathcal{H}_c + u_0\Omega_0 + u_i\Omega_i)$, where the u's are to be treated as Lagrange multipliers. We have the consistency condition, $\dot{\Omega}_0 \approx 0 \Rightarrow \xi \equiv \{H, \pi_0\} = j_0 + \kappa\epsilon^{ij}\partial_i A_j \approx 0$ and $\dot{\Omega}_i \approx 0 \Rightarrow u_i = -\partial_i A_0 + \frac{1}{\kappa}\epsilon_{ik}j_k$. The first is a secondary constraint, the second just relates our Lagrange multiplier to the canonical fields. To check for tertiary constraints, we take $\dot{\xi} \approx 0 \Rightarrow \kappa\epsilon_{ij}\partial_i u_i + \partial_i J_i \approx 0$ which is just a restatement of the above. So there are only three primary constraints and one secondary constraint. Now we classify the constraints further. A first class constraint is one that weakly commutes with all other constraints. Clearly $\Omega_0 \approx 0$ is a first class constraint. Furthermore, $\Omega_i \approx 0$ and $\xi \approx 0$ are second class constraints. We can take a linear combination however,

$$\tilde{\xi} \equiv \xi + \partial_i \Omega_i \tag{50}$$

to form a first class constraint. Thus, we classify:

$$\text{First class constraints :} \quad \Omega_0 \approx 0 \ , \quad \tilde{\xi} \approx 0 \tag{51}$$

$$\text{Second class constraints}: \quad \Omega_i \approx 0. \tag{52}$$

Now we look to form the Dirac bracket. It is defined for two functions f, g in phase space as,

$$\{f,g\}_D \equiv \{f,g\} - \int d^2x\, d^2y\, \{f,\Omega_i(\mathbf{x})\} C^{-1}_{ij}(\mathbf{x},\mathbf{y}) \{\Omega_j(\mathbf{y}),g\} \tag{53}$$

where $C_{ij}(\mathbf{x},\mathbf{y}) \equiv \{\Omega_i(\mathbf{x}),\Omega_j(\mathbf{y})\}$. We compute, $C_{ij}(\mathbf{x},\mathbf{y}) = \kappa\epsilon_{ij}\delta(\mathbf{x}-\mathbf{y})$. Thus, $C^{-1}_{ij} = \frac{1}{\kappa}\epsilon_{ij}\delta(\mathbf{x}-\mathbf{y})$. So,

$$\{\phi(\mathbf{x}),\pi(\mathbf{y})\}_D = \delta(\mathbf{x}-\mathbf{y})\ , \qquad \{\phi^*(\mathbf{x}),\pi^*(\mathbf{y})\}_D = \delta(\mathbf{x}-\mathbf{y}) \tag{54}$$

$$\{A_0(\mathbf{x}),\pi_0(\mathbf{y})\}_D = -\delta(\mathbf{x}-\mathbf{y}) \tag{55}$$

which are the same as the Poisson brackets and,

$$\{A_i(\mathbf{x}),A_j(\mathbf{y})\}_D = -\frac{1}{\kappa}\epsilon_{ij}\delta(\mathbf{x}-\mathbf{y}). \tag{56}$$

B.2 Light-Front Quantization

We use the same Lagrangian but now look at quantizing the theory in the light-cone gauge [12]. For this, it is easiest to rewrite our theory in light-cone coordinates. We perform quantization in the light-cone gauge, namely $A_- = 0$. The canonical momenta are,

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_+\varphi)} = (D_-\varphi)^* \ , \qquad \pi^* = \frac{\partial\mathcal{L}}{\partial(\partial_+\varphi^*)} = D_-\varphi \tag{57}$$

$$\pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_+ A_\mu)} = a\epsilon^{+\mu\nu}A_\nu \tag{58}$$

for $\kappa = 2a$. We have the natural quantity $j_\mu = ie(\varphi^* D_\mu\varphi - \varphi D^\dagger_\mu\varphi)$, which is the conserved (covariant) current. The canonical Hamiltonian is,

$$\mathcal{H}_c = (D_1\varphi)^*(D_1\varphi) - A_+\Omega \tag{59}$$

where

$$\Omega = ie(\pi\varphi - \pi^*\varphi^*) + a\epsilon^{+ij}\partial_i A_j + \partial_i\pi^i. \tag{60}$$

We proceed via the Dirac method for quantization, as done previously.

C Some field theory

Here we give a treatment of classical and quantum field theory which will allow the reader to follow the above analysis. One can view general classical mechanics from two pictures: Lagrangian or Hamiltonian mechanics, both of which are equivalent. For a more extensive treatment, one can refer to [1].

In the Hamiltonian formalism we start with a configuration space (p, q) that describes the momentum and coordinates of our particle and a Hamiltonian H which governs the time evolution of trajectories. The equations of motion are given by,

$$\dot{q} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q}. \tag{61}$$

Equivalently, we can introduce the Poisson bracket $\{,\}_{PB}$ which acts on the phase space functions to write $\dot{q} = \{q, H\}$ and $\dot{p} = \{p, H\}$. In fact, for any function defined on (q, p, t) we have $\dot{F} = \{F, H\} + \partial_t F$, showing more explicitly that the Hamiltonian, in fact, generates the dynamics of the system.

In the Lagrangian formalism we are concerned with the space $(q, \dot{q})$ and the quantity called the action functional $S[q, \dot{q}] = \int dt\ L[q, \dot{q}]$ where L is the Lagrangian that describes the physics of the system in the following way. The actual trajectory that a particle will follow is that which minimizes the action. This leads to the equations of motion,

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}. \tag{62}$$

In the Lagrangian picture we can define the conjugate momentum $p = \frac{\partial L}{\partial \dot{q}}$ and we can perform the Legendre transformation $H = p\dot{q} - L$ to recover the particle's Hamiltonian. In this way, classically, we have a natural equivalence of these two pictures.

In field theory we want to generalize our coordinates to fields which are functions of the coordinates. For this treatment we will be concerned with a scalar field $\varphi(x)$ which is a function on the manifold $x^\mu = (x^0, x^i) = t, \mathbf{x})$ for $i = 1, \ldots, d$. The reader should be aware, however, that the scalar field is not the only type of field. Generally, fields

will be labelled by the way they transform under Lorentz transformation. For instance a spin-$\frac{1}{2}$ field (analogous to a fermion) will transform as a spinor and a spin-1 field will transform as a vector. The generalization of the kinetic term $\frac{1}{2}m\dot{x}^2$ in field theory is the term $\frac{1}{2}\dot{\varphi}^2$ so that the kinetic part of the action is $S_{kin} = \int dx^0 \ \frac{1}{2}\dot{\varphi}^2$. In field theory, however, we are interested in not the Lagrangian L, but the Lagrangian density $\mathcal{L}$ defined by $L = \int d^dx \ \mathcal{L}$. For physical systems the Lagrangian density must be invariant under Lorentz transformations and thus the kinetic part has the form,

$$\mathcal{L}_{kin} = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\nabla\varphi \cdot \nabla\varphi = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi. \tag{63}$$

A free scalar of mass m is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m\phi^2. \tag{64}$$

In general, of course, the Lagrangian will be a general function of fields $\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu\varphi)$. The action takes the form,

$$S = \int_{[t_1,t_2]} dt \ \int d^dx \ \mathcal{L}. \tag{65}$$

Just as in point particle dynamics we can vary the action to get the equations of motion,

$$\frac{\partial\mathcal{L}}{\partial\varphi} = \partial_\mu \frac{\partial\mathcal{L}}{\partial[\partial_\mu\varphi]}. \tag{66}$$

Consider the infinitesimal transformation $\varphi(x) \to \varphi(x) + \delta\varphi(x)$. In terms of the variation parameters suppose $\delta\varphi = \epsilon^i h_i(\varphi)$ where i is the index for the symmetry transformation. If the Lagrangian $\mathcal{L}$ turns out to be invariant under such transformations then we can define the current (density), j_i^μ by,

$$\epsilon^i j_i^\mu \equiv \frac{\partial\mathcal{L}}{\partial[\partial_\mu\varphi]}\delta\varphi \tag{67}$$

which is conserved: $\partial_\mu j^\mu = 0$. As in electrodynamics, we write the current density as $j_i^\mu = (c\rho, \mathbf{j})$ and the conservation equation is just the continuity equation $\partial_t\rho + \nabla \cdot \mathbf{j} = 0$.

To quantize such theories we may proceed in various ways. The first is through the canonical quantization where the classical fields φ are promoted to operators in some

Hilbert space and in the simplest cases, the classical Poisson brackets change to commutators: $\{,\}_{PB} \to i[,]$. (For some systems, usually gauge theories, with constraints, a more general approach is necessary). For this method we are primarily interested in the Hamiltonian density. The other method is through the path integral. For this we are interested in the action of the theory, and in particular, the exponential of the action.

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