The Thermodynamics of Nonlinear Bolometers Near Equilibrium

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Mather realized that Callen & Welton can give nonsensical results for bolometers (including negative noise). Intuitively, he replaced it with an ansatz:

- Model electrical noise as a Gaussian Johnson voltage noise source in series with bolometer with noise PSD based on total resistance
- Work done by or on Johnson noise is coupled to thermal system.

The nonequilibrium part of Mather’s theory is based on an ansatz.

Mather’s unstated ansatz has been very useful, but it is not rigorous.
Existing (e.g. Mather’s) first-principles calculations of thermodynamic noise in resistive bolometers assume:

- linear dissipative elements
- equilibrium Johnson voltage noise
- Gaussian noise sources

In contrast, the following solution is a thermodynamically correct calculation of noise that is rigorous for:

- a simple Markovian bolometer (no hidden variables)
- with linear and quadratically nonlinear $R$ and $G$
- deviating from equilibrium to 1$^{\text{st}}$ and 2$^{\text{nd}}$ order
We solve a thermodynamically closed model with a steady-state nonequilibrium solution: a voltage bias provided by an infinite capacitor, or a current bias provided by an infinite inductor.

- Total energy is constant
- Entropy always increases
- The microscopic physical processes are time reversible

\[
\begin{align*}
U(T) & \quad \text{Bolometer thermal energy} \\
\frac{dQ}{dt} & \quad \text{Heat flowing to the bath} \\
\frac{dJ}{dt} & \quad \text{Joule power dissipation}
\end{align*}
\]

**Constant current bias (e.g. thermistor)**

**Thermal circuit**

\[
U(T) \quad G \quad dJ/dt \quad dQ/dt \quad T_0
\]

**Electrical circuit**

\[
R \quad + \quad C \quad I \quad L \rightarrow \infty
\]

**Constant voltage bias (e.g. TES)**

**Thermal circuit**

\[
c(T) = \frac{dU(T)}{dT}
\]

**Electrical circuit**

\[
\Phi = LI \\
R(I, \delta T) \quad G(\delta T) \quad dQ/dt \quad dJ/dt \quad T_0
\]

\[
q = CV_{\text{ext}} \\
C \rightarrow \infty
\]
Nonequilibrium thermodynamics

Equilibrium thermodynamics

\[
\begin{align*}
\langle I \rangle &= 0 \\
\langle \delta T \rangle &= 0 \\
\delta T &= T - T_0
\end{align*}
\]

• In equilibrium, Mather’s bolometer theory is rigorous

• Electrical noise is Johnson noise

• Thermal noise is “phonon noise” or TFN “thermal fluctuation noise”

\[
S_V = 4k_B TR, \quad S_P = 4k_B T^2 G
\]

(One-sided power spectral density (PSD))

1st and 2nd-order nonequilibrium thermodynamics

Third-order deviations dropped

\[
\begin{align*}
I^3 &\rightarrow 0 \\
\delta T^3 &\rightarrow 0 \quad \text{etc.} \\
I^2 \delta T &\rightarrow 0
\end{align*}
\]

• Mather’s bolometer theory is not rigorous

• Electrical noise is ???

• Thermal noise is ???

\[
S_V = 4k_B TR, \quad S_P = 4k_B T^2 G
\]
Nonlinear thermodynamics

Quadratic nonlinearity

We solve for linear and quadratic dissipative elements (the electrical resistance and the thermal conductance). Higher orders of nonlinearity are dropped.

\[
R(I, \delta T) \equiv \frac{V}{I} \approx \rho_1 + \rho_2 I + \zeta \delta T
\]

\[
G(\delta T) \equiv \frac{dQ/dt}{\delta T} \approx \gamma_1 + \gamma_2 \delta T
\]
The differential equations describing this system are:

**Kirchhoff’s voltage rule**

\[ L \frac{dI}{dt} = -\rho_1 I - \rho_2 I^2 - \zeta I \delta T - \frac{q}{C} \]

**Definition of current**

\[ \frac{dq}{dt} = I \]

**Energy conservation**

\[ \frac{dU}{dt} = -\gamma_1 \delta T - \gamma_2 \delta T^2 - I \frac{d\Phi}{dt} - \frac{q}{C} \frac{dq}{dt} \]
• In a *canonical ensemble*, a thermodynamic system is in equilibrium with an external heat bath, and there is one external temperature that does not fluctuate.

• In a *microcanonical ensemble*, the heat bath is internal to the system, so there can be more than one temperature in the system. Both are equivalent for a large enough system. For convenience, here we use a microcanonical ensemble.

The entropy change $dS$ of the microcanonical ensemble

$$dS = \left( \frac{1}{T_0} - \frac{1}{T} \right) dQ + \frac{dJ}{T}$$

$$dJ = -\frac{\Phi}{L} d\Phi - \frac{q}{C} dq$$

$$dQ = dJ - dU$$

$$dS = \left( \frac{1}{T} - \frac{1}{T_0} \right) dU - \frac{1}{T_0} \frac{\Phi}{L} d\Phi - \frac{1}{T_0} \frac{q}{C} dq$$
Thermodynamic forces

In a microcanonical ensemble, the thermodynamic forces are the partial derivatives of the entropy

\[ X_\alpha = -\frac{\partial S}{\partial x_\alpha} \quad \alpha = \Phi, q, U \]

The state variables \( x_\Phi \equiv \Phi = LI \)
\( x_q \equiv q \)
\( x_U \equiv U \)

The conjugate thermodynamic forces

\[ X_\Phi = \frac{\Phi}{T_0 L} = \frac{I}{T_0} \]
\[ X_q = \frac{q}{T_0 C} = \frac{V_{ext}}{T_0} \]
\[ X_U = \frac{1}{T_0} - \frac{1}{T} = \frac{\delta T}{T_0 T} \]
System differential equation

\[
L \frac{dI}{dt} = -\rho_1 I - \rho_2 I^2 - \zeta I \delta T - \frac{q}{C}
\]
\[
\frac{dq}{dt} = I
\]
\[
\frac{dU}{dt} = -\gamma_1 \delta T - \gamma_2 \delta T^2 - I \frac{d\Phi}{dt} - \frac{q}{C} \frac{dq}{dt}
\]

Already derived

\[
\frac{d\Phi}{dt} = -\rho_1 T_0 X_\Phi - \rho_2 T_0^2 X_\Phi^2 - T_0 X_q - \zeta T_0^3 X_\Phi X_U
\]
\[
\frac{dq}{dt} = T_0 X_\Phi
\]
\[
\frac{dU}{dt} = T_0^2 X_\Phi^2 \rho_1 - \gamma_1 T_0^2 X_U - (\gamma_1 T_0^3 + \gamma_2 T_0^4) X_U^2
\]

Same equations in terms of “thermodynamic forces”; 2\textsuperscript{nd} order terms dropped

\[
\frac{dx_\alpha}{dt} \approx L_{\alpha,\beta} X_\beta + \frac{1}{2} L_{\alpha,\beta,\gamma} X_\beta X_\gamma
\]

L parameters
Onsager reciprocal relations

- Doctoral dissertation on *Onsager reciprocal relations* at Norwegian Institute of Technology was deemed insufficient for a PhD.
- Second try at dissertation (solution of Mathieu equation) deemed incomprehensible by chemistry faculty. Left without PhD
- Faculty position at Johns Hopkins – fired in 1928 for incomprehensible teaching.
- Teaching position at Brown – similarly incomprehensible; fired in 1933.
- Hired at Yale in 1933 – scandal ensued, when it was discovered that he had no PhD.
- Finally granted PhD in 1935.
- Onsager relations received the 1968 Nobel Prize in Chemistry.
- Ended his career and life as a Distinguished University Professor at the University of Miami.

\[ L_{\alpha,\beta} = \epsilon_{\alpha} \epsilon_{\beta} L_{\beta,\alpha} \]

Time-reversal parity of the state variables added by Casimir

\[ \epsilon_{\Phi} = -1 \]
\[ \epsilon_{q} = 1 \]
\[ \epsilon_{U} = 1 \]
Do we satisfy Onsager?

\[ L_{\alpha,\beta} = \epsilon_{\alpha} \epsilon_{\beta} L_{\beta,\alpha} \]
\[ \epsilon_\Phi = -1 \]
\[ \epsilon_q = 1 \]
\[ \epsilon_U = 1 \]

\[ L_{q,\Phi} = -L_{\Phi,q} = T_0 \]

\[ \frac{d\Phi}{dt} = -\rho_1 T_0 X_\Phi - \rho_2 T_0^2 X_\Phi^2 - T_0 X_q - \zeta T_0^3 X_\Phi X_U \]
\[ \frac{dq}{dt} = T_0 X_\Phi \]
\[ \frac{dU}{dt} = T_0^2 X_\Phi^2 \rho_1 - \gamma_1 T_0^2 X_U - (\gamma_1 T_0^3 + \gamma_2 T_0^4) X_U^2 \]
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\[ \frac{dq}{dt} = T_0 X_\Phi \]
\[ \frac{dU}{dt} = T_0^2 X_\Phi^2 \rho_1 - \gamma_1 T_0^2 X_U - (\gamma_1 T_0^3 + \gamma_2 T_0^4) X_U^2 \]
Noise is characterized by correlations

\[ \alpha = \Phi, q, U \]

- Onefold correlators: offset values
  \[ K_\alpha \equiv \lim_{\tau \to 0} (\tau^{-1} \langle \Delta x_\alpha \rangle) \]
- Twofold correlators: power spectral density; Gaussian noise
  \[ K_{\alpha\beta} \equiv \lim_{\tau \to 0} (\tau^{-1} \langle \Delta x_\alpha \Delta x_\beta \rangle) \]
- Threefold correlators: non-Gaussian noise
  \[ K_{\alpha\beta\gamma} \equiv \lim_{\tau \to 0} (\tau^{-1} \langle \Delta x_\alpha \Delta x_\beta \Delta x_\gamma \rangle) \]
Stratonovich’s fluctuation-dissipation relations are derived solely from these few assumptions:

- Markovian processes
- Time-reversal symmetry
- Consistency with equilibrium thermodynamics
- If we drop terms higher than the quadratic, we are only rigorous near equilibrium.

Here is my edited version of Stratonovich’s fluctuation dissipation relations for the correlators appropriate for this microcanonical solution, carried out to 2\textsuperscript{nd} order expansion:

\[
K_\alpha = -L_{\alpha,\beta} \frac{\partial S}{\partial x_\beta} + \frac{1}{2} L_{\alpha,\beta \gamma} \frac{\partial S}{\partial x_\beta} \frac{\partial S}{\partial x_\gamma} + \frac{1}{2} k_B L_{\alpha,\beta \gamma} \frac{\partial^2 S}{\partial x_\beta \partial x_\gamma}
\]

\[
K_{\alpha\beta} = -k_B \left( L_{\alpha,\beta} + L_{\beta,\alpha} \right) - k_B \left( \varepsilon_{\alpha \beta} \varepsilon_{\gamma \delta} L_{\gamma,\alpha\beta} - L_{\alpha,\beta \gamma} - L_{\beta,\alpha \gamma} \right) \frac{\partial S}{\partial x_\gamma}
\]

\[
K_{\alpha\beta\gamma} = k_B^2 \left( 1 - \varepsilon_{\alpha \beta} \varepsilon_{\gamma \delta} \right) \left( L_{\alpha,\beta \gamma} + L_{\beta,\alpha \gamma} + L_{\gamma,\alpha \beta} \right)
\]

The calculation can be carried out to arbitrarily high orders of expansion, valid further from equilibrium, but...

- Higher order calculations have dissipationally undeterminable parameters
- A microscopic theory is needed for accuracy further from equilibrium

The full solution for the bolometer

\[ K_q = \frac{\Phi}{L} = I \]

\[ K_\Phi = -\frac{q}{C} - \rho_1 I - \rho_2 I^2 - \zeta I \delta T + k_B T_0 \frac{\rho_2}{L} \]

\[ K_U = \rho_1 I^2 - \gamma_1 \delta T - \gamma_2 \delta T^2 + k_B T_0 \left( -\frac{\rho_1}{L} + \frac{\gamma_1 + \gamma_2 T_0}{c(T)} \right) \]

\[ K_{\Phi\Phi} = 2k_B T \rho_1 + 6k_B T_0 \rho_2 I + 2k_B T_0 \zeta \delta T \]

\[ K_{UU} = 2k_B T_0 (\gamma_1 T + \gamma_2 T_0 \delta T) \]

\[ K_{U\Phi} = K_{\Phi U} = -2k_B T_0 \rho_1 I \]

\[ K_{\Phi\Phi\Phi} = -12(k_B T_0)^2 \rho_2 \]

Only assumptions

- Markovian processes
- Time-reversal symmetry
- Consistency with equilibrium thermodynamics
- Drop terms higher than quadratic
The onefold correlators describe a rigorous, nonequilibrium steady state solution as $C \to \infty$

Onefold correlators: offset values

- Charge correlator is offset current $K_q = \frac{\Phi}{L} = I$
- Flux correlator is offset voltage $K_\Phi = -\frac{q}{C} - \rho_1 I - \rho_2 I^2 - \zeta I \delta T + k_B T_0 \frac{\rho_2}{L}$
- Energy correlator is power $K_U = \rho_1 I^2 - \gamma_1 \delta T - \gamma_2 \delta T^2 + k_B T_0 \left( -\frac{\rho_1}{L} + \frac{\gamma_1 + \gamma_2 T_0}{c(T)} \right)$
The onefold correlators describe a rigorous, nonequilibrium steady state solution as $C \to \infty$

**Onefold correlators: offset values**

- Charge correlator is offset current
  $$K_q = \frac{\Phi}{L} = I$$

- Flux correlator is offset voltage
  $$K_{\Phi} = -\frac{q}{C} - \rho_1 I - \rho_2 I^2 - \zeta I \delta T + k_B T_0 \frac{\rho_2}{L}$$

- Energy correlator is power
  $$K_U = \rho_1 I^2 - \gamma_1 \delta T - \gamma_2 \delta T^2 + k_B T_0 \left( -\frac{\rho_1}{L} + \frac{\gamma_1 + \gamma_2 T_0}{c(T)} \right)$$

**Interesting term:**

Prevents the rectification of noise, eliminating a perpetual motion machine
The two correlators describe Gaussian noise

Twofold correlators: noise PSD

- Nonequilibrium “Johnson” noise
  \[ K_{\Phi \Phi} = 2k_B T \rho_1 + 6k_B T_0 \rho_2 I + 2k_B T_0 \zeta \delta T \]
- Nonequilibrium “phonon” noise
  \[ K_{UU} = 2k_B T_0 (\gamma_1 T + \gamma_2 T_0 \delta T) \]
- Electrothermal feedback
  \[ K_{U\Phi} = K_{\Phi U} = -2k_B T_0 \rho_1 I \]
Twofold correlators: noise PSD

- Nonequilibrium “Johnson” noise
  \[ K_{\Phi \Phi} = 2k_B T \rho_1 + 6k_B T_0 \rho_2 I + 2k_B T_0 \zeta \delta T \]
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- Electrothermal feedback
  \[ K_{U\Phi} = K_{\Phi U} = -2k_B T_0 \rho_1 I \]

Equilibrium Johnson noise

\[ S_V = 4k_B T R \]
The two correlators describe Gaussian noise.

Twofold correlators: noise PSD

- Nonequilibrium “Johnson” noise
  \[ K_{\Phi\Phi} = 2k_B T \rho_1 + 6k_B T_0 \rho_2 I + 2k_B T_0 \zeta \delta T \]
- Nonequilibrium “phonon” noise
  \[ K_{UU} = 2k_B T_0 (\gamma_1 T + \gamma_2 T_0 \delta T) \]
- Electrothermal feedback
  \[ K_{U\Phi} = K_{U\Phi} = -2k_B T_0 \rho_1 I \]

Equilibrium thermal fluctuation noise

\[ S_P = 4k_B T^2 G \]
And dropping 2nd-order terms:

\[ S_V = 4k_B T \rho_1 + 12k_B T_0 \rho_2 I + 4k_B T_0 \zeta \delta T \]

Note that if \( \delta T \to 0 \), this equation is consistent with Stratonovich’s calculation of the noise in a nonlinear resistor without temperature dependence. Now introducing the usual “alpha” and “beta” parameters for the TES:

\[
\alpha_I \equiv \left. \frac{\partial \log(R)}{\partial \log(T)} \right|_I \\
\beta_I \equiv \left. \frac{\partial \log(R)}{\partial \log(I)} \right|_T
\]

And dropping 2nd-order terms:

\[ S_V = 4k_B T R \left( 1 + 2 \beta_I \right) \]

All the terms dependent on \( \alpha \) cancel, but we are left with “\( \beta \) noise”. This calculation has a more complex result than Mather’s ansatz.
Our equation for the thermal fluctuation noise, or “phonon noise” correlator, is consistent with Mather’s expressions for phonon noise in a diffuse link.

It is also consistent with some expressions for the limit of a specular link.
• There is a threefold correlator

\[ K_{\Phi\Phi} = -12 \left( k_B T_0 \right)^2 \rho_2 \]

• The noise is (very) weakly non-Gaussian

• This isn’t in the equilibrium theory.
Conclusions

• Mather’s classic bolometer theory is a useful, but imperfect ansatz.

• Nonequilibrium thermodynamics provides a rigorous expression for the noise of a simple nonlinear bolometer fairly near to equilibrium. “Fairly” is difficult to define.

• The noise is weakly non-Gaussian

• There is a “β-noise” contribution from nonlinearity close to equilibrium:

  \[
  S_V = 4k_B TR \left( 1 + 2\beta_I \right)
  \]

• Far from equilibrium, the phrase “excess noise” is misleading. There is just “noise” and its ratio to Mather’s (imperfect) ansatz.

• Simple thermodynamics has no prediction for the baseline noise far from equilibrium