The Thermodynamics of Nonlinear Bolometers Near Equilibrium

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Mather's ansatz

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Bolometer noise: nonequilibrium theory

John C. Mather

The nonequilibrium part of Mather's theory is based on an ansatz…

New theoretical results for noise in cryogenic bolometers are derived. Johnson noise is reduced by as much as 60% by electrothermal feedback from the bias supply. Phonon noise in the thermal link is reduced by as much as 30% relative to the usual equilibrium formula. Photon noise in the Rayleigh-Jeans limit is computed with attention to the attenuation of the photon correlations in the light beam. Basic results on bolometer responsivity, time constant, and thermal properties are presented in a new and convenient form. Excess 1/*f* and contact shot noise are also discussed.

Callen & Welton's classical generalization of Nyquist's formula:

$$S_V = 4k_B T \,\mathrm{Re}\{Z(\omega)\}$$

(non-Markov, rigorous in equilibrium)

Mather realized that Callen & Welton can give nonsensical results for bolometers (including negative noise). Intuitively, he replaced it with an *ansatz:*

- Model electrical noise as a Gaussian Johnson voltage noise source in series with bolometer with noise PSD based on *total* resistance

$$S_V = 4k_B T R$$
$$R \equiv V/I$$

- Work done by or on Johnson noise is coupled to thermal system.

Mather's unstated ansatz has been very useful, but it is not rigorous

Nonequilibrium thermodynamics

Existing (e.g. Mather's) first-principles calculations of thermodynamic noise in resistive bolometers assume:

- linear dissipative elements
- equilibrium Johnson voltage noise
- Gaussian noise sources

In contrast, the following solution is a thermodynamically correct calculation of noise that is rigorous for:

- a simple Markovian bolometer (no hidden variables)
- with linear and quadratically nonlinear *R* and *G*
- deviating from equilibrium to 1st and 2nd order

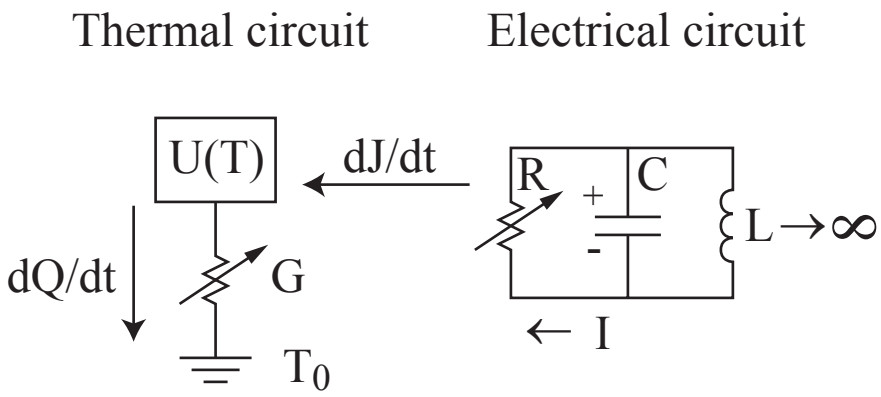
A closed thermodynamic model

We solve a thermodynamically closed model with a steady-state nonequilibrium solution: a voltage bias provided by an infinite capacitor, or a current bias provided by an infinite inductor.

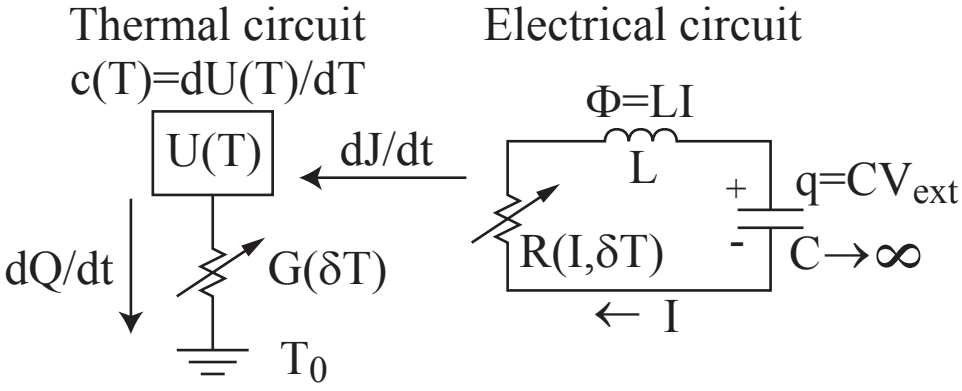
- Total energy is constant
- Entropy always increases
- The microscopic physical processes are time reversible

U(T)	Bolometer thermal energy
dQ/dt	Heat flowing to the bath
dJ/dt	Joule power dissipation

Constant current bias (e.g. thermistor)



Constant voltage bias (e.g. TES)



Nonequilibrium thermodynamics

Equilibrium thermodynamics

$$\langle I \rangle = 0$$

$$\langle \delta T \rangle = 0$$

$$\delta T \equiv T - T_0$$

- In equilibrium, Mather's bolometer theory is rigorous
- Electrical noise is Johnson noise
- Thermal noise is "phonon noise" or TFN "thermal fluctuation noise"

$$S_V = 4k_B TR$$

$$S_P = 4k_B T^2 G$$

(One-sided power spectral density (PSD))

1st and 2nd-order nonequilibrium thermodynamics

Third-order deviations dropped

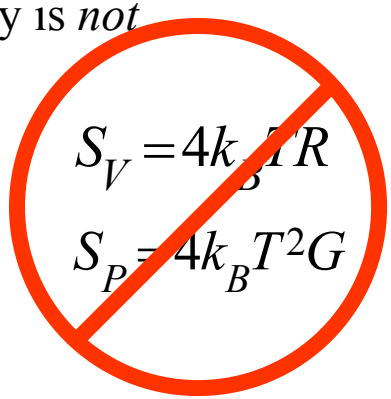
$$I^3 \to 0$$

$$\delta T^3 \to 0$$

$$I^2 \delta T \to 0$$

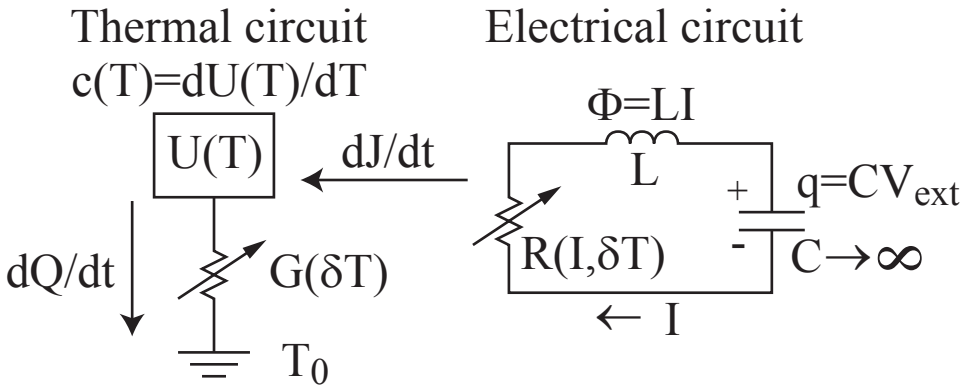
etc.

- Mather's bolometer theory is *not* rigorous
- Electrical noise is ???
- Thermal noise is ???



Quadratic nonlinearity

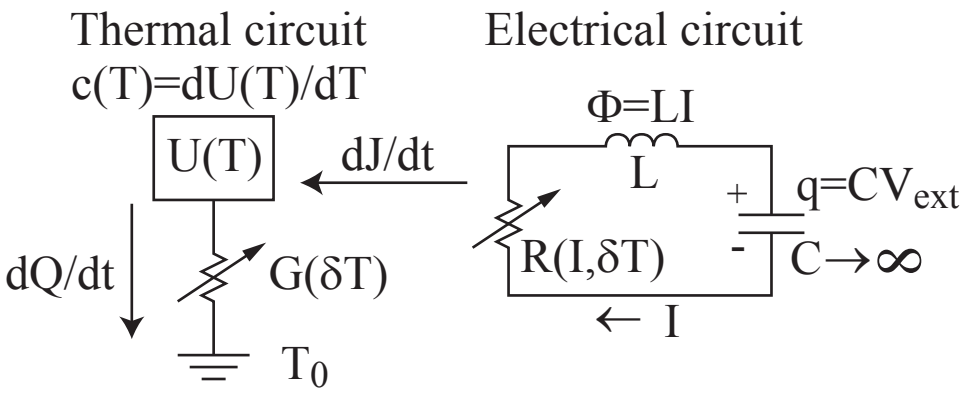
We solve for linear and quadratic dissipative elements (the electrical resistance and the thermal conductance). Higher orders of nonlinearity are dropped.



$$R(I,\delta T) \equiv \frac{V}{I} \approx \rho_1 + \rho_2 I + \zeta \delta T$$

$$G(\delta T) \equiv \frac{dQ/dt}{\delta T} \approx \gamma_1 + \gamma_2 \delta T$$

Thermo-electric differential equations



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$$G(\delta T) \equiv \frac{dQ/dt}{\delta T} \approx \gamma_1 + \gamma_2 \delta T$$

The differential equations describing this system are:

Kirchhoff's voltage rule $$L\frac{dI}{dt} = -\rho_1 I - \rho_2 I^2 - \zeta I \delta T - \frac{q}{C}$$

Definition of current $$\frac{dq}{dt} = I$$

Energy conservation $$\frac{dU}{dt} = -\gamma_1 \delta T - \gamma_2 \delta T^2 - I\frac{d\Phi}{dt} - \frac{q}{C}\frac{dq}{dt}$$

Microcanonical ensemble

- In a *canonical ensemble*, a thermodynamic system is in equilibrium with an external heat bath, and there is one external temperature that does not fluctuate.

- In a *microcanonical ensemble*, the heat bath is internal to the system, so there can be more than one temperature in the system.

Both are equivalent for a large enough system. For convenience, here we use a microcanonical ensemble.

The entropy change dS of the microcanonical ensemble

$$dS = \left(\frac{1}{T_0} - \frac{1}{T}\right) dQ + \frac{dJ}{T}$$

$$dJ = -\frac{\Phi}{L} d\Phi - \frac{q}{C} dq$$

$$dQ = dJ - dU$$

$$dS = \left(\frac{1}{T} - \frac{1}{T_0}\right) dU - \frac{1}{T_0}\frac{\Phi}{L} d\Phi - \frac{1}{T_0}\frac{q}{C} dq$$

Thermodynamic forces

In a microcanonical ensemble, the thermodynamic forces are the partial derivatives of the entropy

$$X_\alpha = -\frac{\partial S}{\partial x_\alpha} \qquad \alpha = \Phi, q, U$$

The state variables → The conjugate *thermodynamic forces*

$$x_\Phi \equiv \Phi = LI$$
$$x_q \equiv q$$
$$x_U \equiv U$$

$$X_\Phi = \frac{\Phi}{T_0 L} = \frac{I}{T_0}$$
$$X_q = \frac{q}{T_0 C} = \frac{V_{ext}}{T_0}$$
$$X_U = \frac{1}{T_0} - \frac{1}{T} = \frac{\delta T}{T_0 T}$$

System differential equation

$$L\frac{dI}{dt} = -\rho_1 I - \rho_2 I^2 - \zeta I \delta T - \frac{q}{C}$$
$$\frac{dq}{dt} = I$$
$$\frac{dU}{dt} = -\gamma_1 \delta T - \gamma_2 \delta T^2 - I\frac{d\Phi}{dt} - \frac{q}{C}\frac{dq}{dt}$$

Already derived

$$\frac{d\Phi}{dt} = -\rho_1 T_0 X_\Phi - \rho_2 T_0^2 X_\Phi^2 - T_0 X_q - \zeta T_0^3 X_\Phi X_U$$
$$\frac{dq}{dt} = T_0 X_\Phi$$
$$\frac{dU}{dt} = T_0^2 X_\Phi^2 \rho_1 - \gamma_1 T_0^2 X_U - (\gamma_1 T_0^3 + \gamma_2 T_0^4) X_U^2$$

Same equations in terms of "thermodynamic forces"; 2nd order terms dropped

$$\frac{dx_\alpha}{dt} \approx L_{\alpha,\beta} X_\beta + \frac{1}{2} L_{\alpha,\beta\gamma} X_\beta X_\gamma$$

L parameters

Onsager reciprocal relations

- Doctoral dissertation on *Onsager reciprocal relations* at Norwegian Institute of Technology was deemed insufficient for a PhD.
- Second try at dissertation (solution of Mathieu equation) deemed incomprehensible by chemistry faculty. Left without PhD
- Faculty position at Johns Hopkins – fired in 1928 for incomprehensible teaching.
- Teaching position at Brown – similarly incomprehensible; fired in 1933.
- Hired at Yale in 1933 – scandal ensued, when it was discovered that he had no PhD.
- Finally granted PhD in 1935.
- Onsager relations received the 1968 Nobel Prize in Chemistry.
- Ended his career and life as a Distinguished University Professor at the University of Miami.

Lars Onsager

$$L_{\alpha,\beta} = \epsilon_\alpha \epsilon_\beta L_{\beta,\alpha}$$

Time-reversal parity of the state variables added by Casimir

$$\epsilon_\Phi = -1$$
$$\epsilon_q = 1$$
$$\epsilon_U = 1$$

Do we satisfy Onsager?

$$L_{\alpha,\beta} = \epsilon_\alpha \epsilon_\beta L_{\beta,\alpha}$$
$$\epsilon_\Phi = -1$$
$$\epsilon_q = 1$$
$$\epsilon_U = 1$$

?

$$L_{q,\Phi} = -L_{\Phi,q} = T_0$$

$$\frac{d\Phi}{dt} = -\rho_1 T_0 X_\Phi - \rho_2 T_0^2 X_\Phi^2 - T_0 X_q - \zeta T_0^3 X_\Phi X_U$$
$$\frac{dq}{dt} = T_0 X_\Phi$$
$$\frac{dU}{dt} = T_0^2 X_\Phi^2 \rho_1 - \gamma_1 T_0^2 X_U - (\gamma_1 T_0^3 + \gamma_2 T_0^4) X_U^2$$

Do we satisfy Onsager?

$$L_{\alpha,\beta} = \epsilon_\alpha \epsilon_\beta L_{\beta,\alpha}$$

$$\epsilon_\Phi = -1$$

$$\epsilon_q = 1$$

$$\epsilon_U = 1$$

$$L_{q,\Phi} = -L_{\Phi,q} = T_0$$

YES

$$\frac{d\Phi}{dt} = -\rho_1 T_0 X_\Phi - \rho_2 T_0^2 X_\Phi^2 - T_0 X_q - \zeta T_0^3 X_\Phi X_U$$

$$\frac{dq}{dt} = T_0 X_\Phi$$

$$\frac{dU}{dt} = T_0^2 X_\Phi^2 \rho_1 - \gamma_1 T_0^2 X_U - (\gamma_1 T_0^3 + \gamma_2 T_0^4) X_U^2$$

Correlators and noise

Noise is characterized by correlations

$$\alpha = \Phi, q, U$$

- Onefold correlators: offset values
- Twofold correlators: power spectral density; Gaussian noise
- Threefold correlators: non-Gaussian noise

$$K_\alpha \equiv \lim_{\tau \to 0} (\tau^{-1} \langle \Delta x_\alpha \rangle)$$

$$K_{\alpha\beta} \equiv \lim_{\tau \to 0} (\tau^{-1} \langle \Delta x_\alpha \Delta x_\beta \rangle)$$

$$K_{\alpha\beta\gamma} \equiv \lim_{\tau \to 0} (\tau^{-1} \langle \Delta x_\alpha \Delta x_\beta \Delta x_\gamma \rangle)$$

Stratonovich's equations

Stratonovich's fluctuation-dissipation relations are derived solely from these few assumptions

- Markovian processes
- Time-reversal symmetry
- Consistency with equilibrium thermodynamics
- If we drop terms higher than the quadratic, we are only rigorous near equilibrium.

R. Stratonovich, *Nonlinear Nonequilibrium Thermodynamics I*, Springer-Verlag, 1992.

Stratonovich's equations

Here is my edited version of Stratonovich's fluctuation dissipation relations for the correlators appropriate for this microcanonical solution, carried out to 2nd order expansion

$$K_\alpha = -L_{\alpha,\beta}\frac{\partial S}{\partial x_\beta} + \frac{1}{2}L_{\alpha,\beta\gamma}\frac{\partial S}{\partial x_\beta}\frac{\partial S}{\partial x_\gamma} + \frac{1}{2}k_B L_{\alpha,\beta\gamma}\frac{\partial^2 S}{\partial x_\beta \partial x_\gamma}$$

$$K_{\alpha\beta} = -k_B\left(L_{\alpha,\beta} + L_{\beta,\alpha}\right) - k_B\left(\varepsilon_\alpha\varepsilon_\beta\varepsilon_\gamma L_{\gamma,\alpha\beta} - L_{\alpha,\beta\gamma} - L_{\beta,\alpha\gamma}\right)\frac{\partial S}{\partial x_\gamma}$$

$$K_{\alpha\beta\gamma} = k_B^2\left(1 - \varepsilon_\alpha\varepsilon_\beta\varepsilon_\gamma\right)\left(L_{\alpha,\beta\gamma} + L_{\beta,\alpha\gamma} + L_{\gamma,\alpha\beta}\right)$$

The calculation can be carried out to arbitrarily high orders of expansion, valid further from equilibrium, but…

- Higher order calculations have *dissipationally undeterminable parameters*
- A microscopic theory is needed for accuracy further from equilibrium

R. Stratonovich, *Nonlinear Nonequilibrium Thermodynamics I*, Springer-Verlag, 1992.

The full solution for the bolometer

$$K_q = \frac{\Phi}{L} = I$$

$$K_\Phi = -\frac{q}{C} - \rho_1 I - \rho_2 I^2 - \zeta I \delta T + k_\mathrm{B} T_0 \frac{\rho_2}{L}$$

$$K_U = \rho_1 I^2 - \gamma_1 \delta T - \gamma_2 \delta T^2 + k_\mathrm{B} T_0 \left(-\frac{\rho_1}{L} + \frac{\gamma_1 + \gamma_2 T_0}{c(T)} \right)$$

$$K_{\Phi\Phi} = 2 k_\mathrm{B} T \rho_1 + 6 k_\mathrm{B} T_0 \rho_2 I + 2 k_\mathrm{B} T_0 \zeta \delta T$$

$$K_{UU} = 2 k_\mathrm{B} T_0 \left(\gamma_1 T + \gamma_2 T_0 \delta T \right)$$

$$K_{U\Phi} = K_{\Phi U} = -2 k_\mathrm{B} T_0 \rho_1 I$$

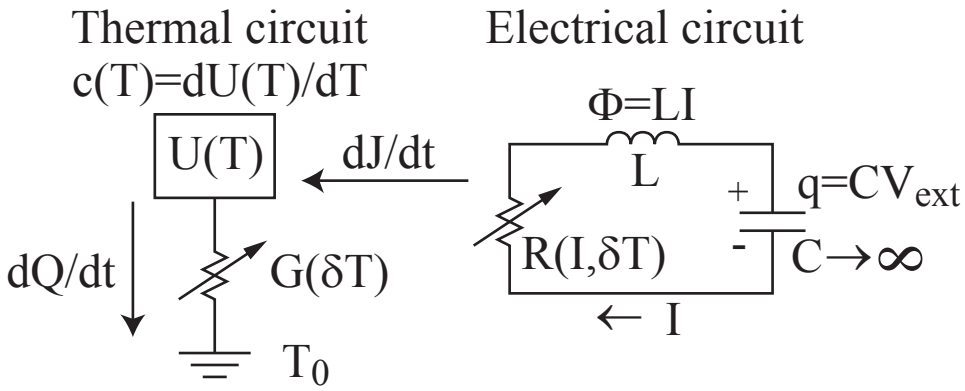
$$K_{\Phi\Phi\Phi} = -12 (k_\mathrm{B} T_0)^2 \rho_2$$

Only assumptions

- Markovian processes
- Time-reversal symmetry
- Consistency with equilibrium thermodynamics
- Drop terms higher than quadratic

Onefold correlators

The onefold correlators describe a rigorous, *nonequilibrium steady state solution* as $C \to \infty$



Onefold correlators: offset values

- Charge correlator is offset current $K_q = \dfrac{\Phi}{L} = I$
- Flux correlator is offset voltage $K_\Phi = -\dfrac{q}{C} - \rho_1 I - \rho_2 I^2 - \zeta I \delta T + k_\mathrm{B} T_0 \dfrac{\rho_2}{L}$
- Energy correlator is power $K_U = \rho_1 I^2 - \gamma_1 \delta T - \gamma_2 \delta T^2 + k_\mathrm{B} T_0 \left(-\dfrac{\rho_1}{L} + \dfrac{\gamma_1 + \gamma_2 T_0}{c(T)} \right)$

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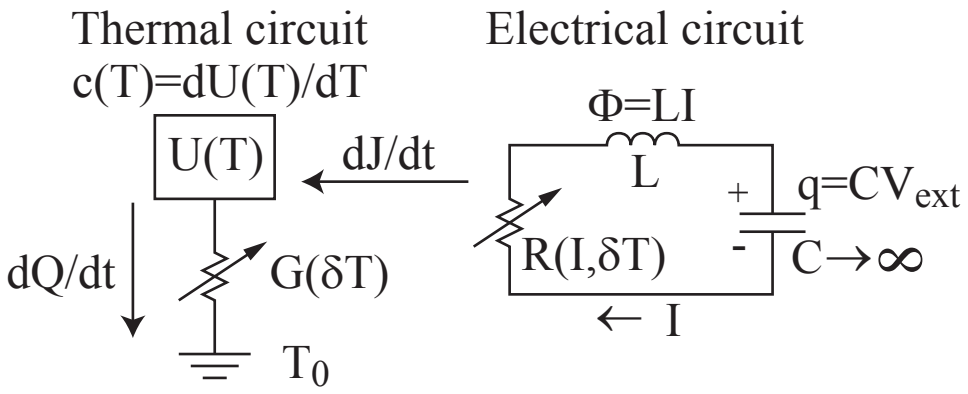

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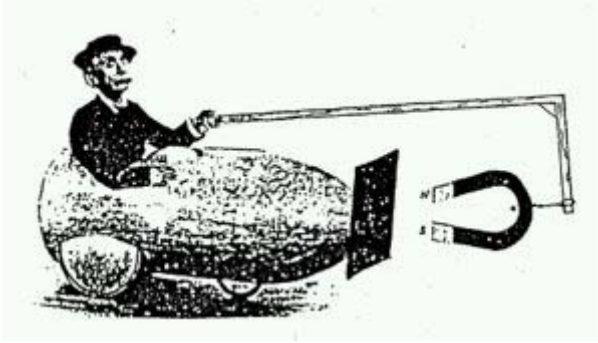
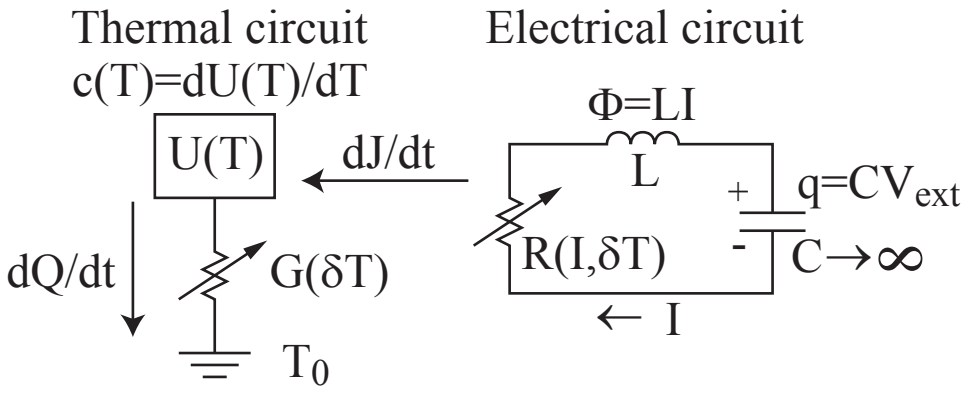
Interesting term:

Prevents the rectification of noise, eliminating a perpetual motion machine



Twofold correlators

The two correlators describe Gaussian noise



Twofold correlators: noise PSD

- Nonequilibrium “Johnson” noise
- Nonequilibrium “phonon” noise
- Electrothermal feedback

$$K_{\Phi\Phi} = 2k_{\mathrm{B}}T\rho_1 + 6k_{\mathrm{B}}T_0\rho_2 I + 2k_{\mathrm{B}}T_0\zeta\delta T$$

$$K_{UU} = 2k_{\mathrm{B}}T_0\left(\gamma_1 T + \gamma_2 T_0\delta T\right)$$

$$K_{U\Phi} = K_{\Phi U} = -2k_{\mathrm{B}}T_0\rho_1 I$$

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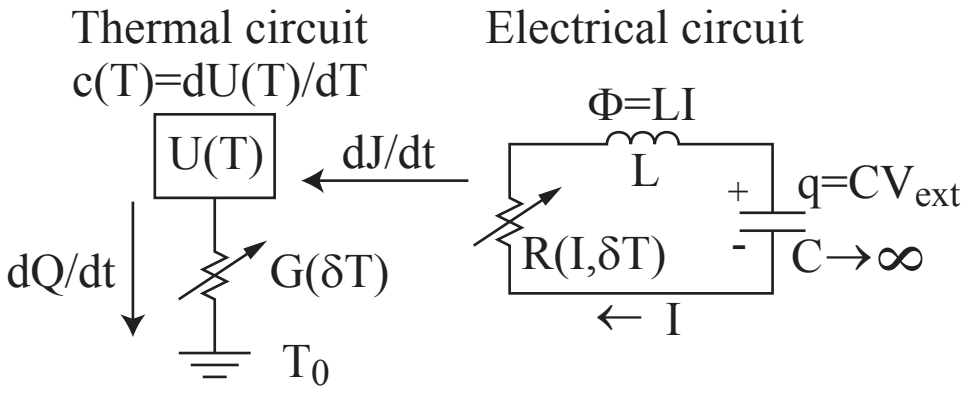

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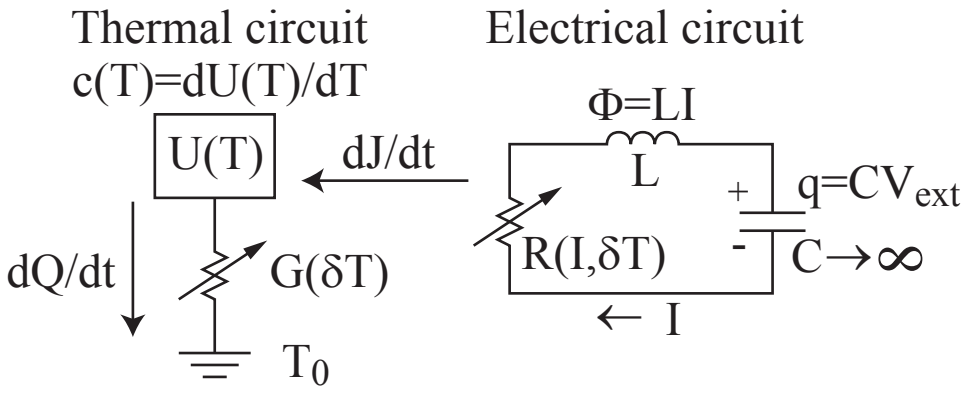
Equilibrium Johnson noise $S_V = 4k_B TR$

Twofold correlators

The two correlators describe Gaussian noise



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$$K_{U\Phi} = K_{\Phi U} = -2k_{\mathrm{B}}T_0\rho_1 I$$

Equilibrium thermal fluctuation noise $S_P = 4k_B T^2 G$

"β noise"

$$S_V = 4k_B T \rho_1 + 12k_B T_0 \rho_2 I + 4k_B T_0 \zeta \delta T$$

Note that if $\delta T \rightarrow 0$, this equation is consistent with Stratonovich's calculation of the noise in a nonlinear resistor without temperature dependence. Now introducing the usual "alpha" and "beta" parameters for the TES:

$$\alpha_I \equiv \left. \frac{\partial \log(R)}{\partial \log(T)} \right|_I$$

$$\beta_I \equiv \left. \frac{\partial \log(R)}{\partial \log(I)} \right|_T$$

And dropping 2nd-order terms:

$$S_V = 4k_B TR\left(1 + 2\beta_I\right)$$

All the terms dependent on α cancel, but we are left with "β noise". This calculation has a more complex result than Mather's ansatz.

Thermal fluctuation noise

$$K_{UU} = 2k_B T_0 \left(\gamma_1 T + \gamma_2 T_0 \delta T \right)$$

$$S_P = 4k_B T_0 \left(\gamma_1 T + \gamma_2 T_0 \delta T \right)$$

•Our equation for the thermal fluctuation noise, or “phonon noise” correlator, is consistent with Mather’s expressions for phonon noise in a diffuse link.

• It is also consistent with some expressions for the limit of a specular link.

Threefold correlator

- There is a threefold correlator

$$K_{\Phi\Phi\Phi} = -12\left(k_B T_0\right)^2 \rho_2$$

- The noise is (very) weakly non-Gaussian
- *This isn't in the equilibrium theory.*

Conclusions

- Mather's classic bolometer theory is a useful, but imperfect *ansatz*.
- Nonequilibrium thermodynamics provides a rigorous expression for the noise of a simple nonlinear bolometer fairly near to equilibrium. "Fairly" is difficult to define.
- The noise is weakly non-Gaussian
- There is a "β-noise" contribution from nonlinearity close to equilibrium: $S_V = 4k_B TR\left(1+2\beta_I\right)$

- Far from equilibrium, the phrase "excess noise" is misleading. There is just "noise" and its ratio to Mather's (imperfect) ansatz.
- Simple thermodynamics has *no prediction* for the baseline noise far from equilibrium

Paper: http://qdev.boulder.nist.gov/817.03/pubs/downloads/qsensors/IrwinNonequilibriumBolometers.pdf