Coulomb's Law

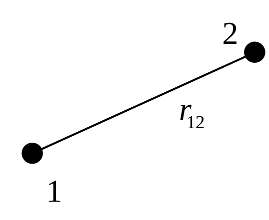
Disclaimer: These lecture notes are not meant to replace the course textbook. The content may be incomplete. Some topics may be unclear. These notes are only meant to be a study aid and a supplement to your own notes. Please report any inaccuracies to the professor.

Electric Charge

- It is an intrinsic property of particles (i.e. electrons and protons)
- Comes in both positive and negative amounts (assignment of + and – chosen by Ben Franklin)
- Usually denote charge by letter "q", unit of measure is the Coulomb, C, in SI units
  - Electron: $q_e = -e = -1.6022 \times 10^{-19}$ C
  - Proton: $q_p = e = 1.6022 \times 10^{-19}$ C
  - In fact, all charge is quantized in integer multiples of "e" (see further below)
- Most matter is electrically neutral (balanced: equal amounts + and –)
  - For example, hydrogen, as with all atoms, is neutral. That is lucky for us, otherwise we would have strong attractions to other pieces of matter. But this observation is not explained by any verifiable theory yet!
- Can get a net imbalance of electric charge:
  - Silk on glass ⇒ excess + charge on glass
  - Fur on plastic ⇒ excess – charge on plastic
- Net charge is always conserved
- Like-sign charges repel
- Opposite-sign charges attract
- Need a force law to describe this!

Force Laws

Unit of force is Newtons, N, (kg m s^{-2}), in SI units



Newton's Law of Gravity:

$$\mathbf{F}_{grav} = G\frac{m_1 m_2}{r_{12}^2}\hat{\mathbf{r}}_{12}$$

m is mass, measured in kg (SI units). Think of it as the "charge" for gravity.
$G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} is Newton's gravitational constant
r_{12} is the distance between two masses, i.e. $|\mathbf{r}_{12}|$
$\hat{\mathbf{r}}_{12}$ is a unit vector pointing along the direction between mass 1 and mass 2

Note that mass is always defined positive (only one type of gravitational "charge"). Also, the force is always attractive, not repulsive. So the direction of the force is always toward another mass.

Coulomb's Law (Law of Electrostatics, 1785):

$$\mathbf{F}_{coul} = K\frac{q_1 q_2}{r_{12}^2}\hat{\mathbf{r}}_{12}$$

q is electric charge, measured in C (SI units)

$K = \frac{1}{4\pi\varepsilon_0} = 9\times10^9$ N m^2 C^{-2} is the electrostatic constant [Note: some books use "k"]

$\varepsilon_0 = 8.85\times10^{-12}$ C^2 N^{-1} m^{-2} is the *electric permittivity* constant

Note that electric charge q can be positive *or* negative. The force is either attractive (opposite charges) or repulsive (like-sign charges). So the direction of the force is either toward another charge (attractive) or oppositely directed from another charge (repulsive). That is, the force is always aligned along $\hat{\mathbf{r}}_{12}$. Use this guidance in determining the direction of a force along a particular axis, not the sign of $q_1 \times q_2$ directly.

Interpretation of force:

A force causes an object to accelerate if it is free to move.

Newton's Second Law: $\mathbf{F} = m\mathbf{a}$

So for the Coulomb force acting on two charged particles otherwise free to move, the acceleration of one of the particles will be:

$$\mathbf{a}_1 = \frac{\mathbf{F}_1}{m_1} = \frac{K}{m_1}\frac{q_1 q_2}{r_{12}^2}\hat{\mathbf{r}}_{12}$$

Comparison of Gravitational and Electric Forces:

Compare strengths of forces for two objects separated by 1m. Each object has a mass of 1 kg and a charge of 1 C:

$$|F_{grav}| = G\frac{1\cdot1}{1^2} = 6.67\times10^{-11}\text{ N}$$

$$|F_{coul}| = K\frac{1\cdot1}{1^2} = 9\times10^{9}\text{ N} \qquad > 10^{20}\times|F_{grav}|\ !$$

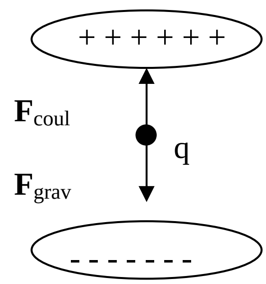
Compare attractive force between electron and proton in hydrogen:

$$|F_{grav}| = G\frac{m_e m_p}{a_0^2} = 6.67\times10^{-11}\frac{\left(9.11\times10^{-31}\text{ kg}\right)\left(1.67\times10^{-27}\text{ kg}\right)}{\left(0.5\times10^{-10}\text{ m}\right)^2} = 4\times10^{-47}\text{ N}$$

$$|F_{coul}| = K\frac{q_e q_p}{a_0^2} = \left(9\times10^{9}\text{ N}\right)\frac{\left(1.6\times10^{-19}\text{ C}\right)^2}{\left(0.5\times10^{-10}\text{ m}\right)^2} = 9\times10^{-8}\text{ N} \qquad > 10^{40}\times|F_{grav}|\ !$$

Electric Charge Quantization

Experiment done by American physicist Robert Millikan demonstrated that electric charge is quantized.



Millikan's oil drop experiment $\Rightarrow$
balanced gravitational force, $\mathbf{F}_{grav}$, with electric force, $\mathbf{F}_{coul}$

$$\Rightarrow q = n\cdot e \qquad n = 0, \pm1, \pm2, \ldots \qquad e = 1.6022\times10^{-19}\text{ C}$$

There exists an elementary unit of charge!
No smaller charge observed, although "quarks" (constituents of protons and neutrons) are expected to have fractional electric charges. But nevertheless, quantization is still a unique feature. Electrons: $q = -e$, protons: $q = +e$. We don't know why balanced!

1 Coulomb of electrons is $\dfrac{1\text{ C}}{1.6\times10^{-19}\text{ C}} = 6\times10^{18}$ electrons !

Electric Charge Conservation

The net sum of electric charge is always conserved. So when a charged conducting object is brought into contact with another conducting object, the charges in the two objects may redistribute, but the net charge of the combined two-object system will remain the same. Likewise, charge is always conserved in reactions:

$$p^+ + e^- \to {}^1_1H$$
$$e^+ + e^- \to \gamma\gamma$$
$${}^{238}_{92}U \to {}^{234}_{90}Th + {}^4_2H$$

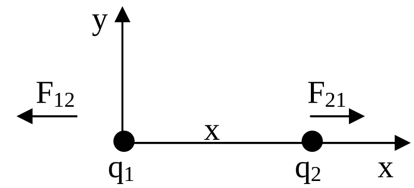
Coulomb's Law Example (1-dimension)

Consider 2 point charges, q_1 and q_2, separated by a distance x.
Let:

$q_1 = -3q \quad q > 0$ (q is some positive number)

$q_2 = -q$ (both charges are negative)

$x = 1$ m



Convention:

$\vec{\mathbf{F}}_{12}$ denotes the force acting on particle 1 from the presence of particle 2

Since the force is repulsive (same-sign electric charges),

$\vec{\mathbf{F}}_{12} = -F_{12}\hat{\mathbf{x}}$ (points in negative x direction).

By Newton's 3rd Law, that for every action (force) there is an equal and opposite reaction, the force on particle 2 from particle 1 is:

$\vec{\mathbf{F}}_{21} = -\vec{\mathbf{F}}_{12} = F_{12}\hat{\mathbf{x}}$ (points in positive x direction).

Again, use this line of reasoning to determine the direction of the force and not the sign in Coulomb's Law.

The magnitude of the force is given by Coulomb's Law:

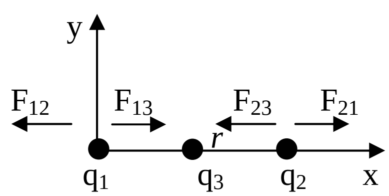
$$F_{12} = |\mathbf{F}_{12}| = K\frac{|q_1||q_2|}{x^2} = 3Kq^2$$

Since the force is non-zero and repulsive, the charges will accelerate in the directions specified by the forces.

Can we add a third charge to counteract this force and leave all charge stationary?
Yes! Place a third <u>positive</u> charge between charges 1 and 2.

<u>Note that the superposition principle holds</u>: adding a third charge does not affect the force between charges 1 and 2.

Let particle 1 reside at $x = 0$, particle 2 at $x = 1$ m, and particle 3 at $x = r$, where $0 < r < 1$.



Find r such that all charges remain at rest, and determine q_3 $\Rightarrow$ 2 unknowns.

Equilibrium $\Rightarrow \sum_i \mathbf{F}_{ji} = 0$ Sum of all forces acting on particle *j* equals 0.

For particle 1 this means: $\mathbf{F}_{12} + \mathbf{F}_{13} = 0$ or $\mathbf{F}_{12} = -\mathbf{F}_{13}$

So:

$$K\frac{q_1 q_2}{1^2} = -K\frac{q_1 q_3}{r^2} \quad r = \text{separation between 1 and 3}$$

$$\Rightarrow q_3 = -r^2 q_2$$

Equilibrium of particle 2 implies: $\mathbf{F}_{21} + \mathbf{F}_{23} = 0$ or $\mathbf{F}_{21} = -\mathbf{F}_{23}$

$$K\frac{q_2 q_1}{1^2} = -K\frac{q_2 q_3}{(1-r)^2} \quad 1-r = \text{separation between 2 and 3}$$

$$\Rightarrow q_3 = -(1-r)^2 q_1$$

Setting the two equations we obtained for q_3 equal yields:

$$r^2 q_2 = (1-r)^2 q_1$$

Plug in:

$$q_1 = -3q$$

$$q_2 = -q$$

$$\Rightarrow qr^2 = (1-2r+r^2)3q$$

$$\Rightarrow 2r^2 - 6r + 3 = 0$$

quadratic equation $\Rightarrow r = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$r = \frac{6 \pm \sqrt{36-4\cdot 2\cdot 3}}{2\cdot 2} = \frac{3}{2} \pm \frac{1}{2}\sqrt{3}$$

Only the solution $0 < r < 1$ works, so $r = \frac{1}{2}\left(3-\sqrt{3}\right) \simeq 0.63$ (a little more than half-way)

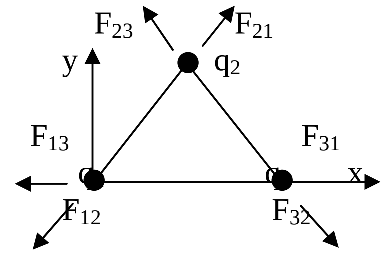
We can solve for q_3:

$$q_3 = -r^2 q_2 \qquad q_2 = -q$$

$$\Rightarrow q_3 \simeq +0.4q \quad \text{indeed positive}$$

This is an unstable equilibrium. If any charge is moved, the forces will no longer balance and will tend to move the charges even farther away from the equilibrium point (i.e. there is no restoring force to make the equilibrium stable).

Coulomb's Law Example (2-dimensions)



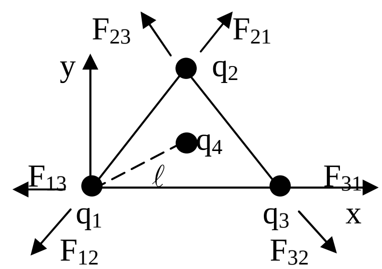
Consider 3 charges placed in the shape of an equilateral triangle. Each vertex carries an equal charge: $q_1 = q_2 = q_3 = -q$ (negative) and the side length is *d*.

Forces on each charge are repulsive (triangle wants to expand). Magnitude of forces are all equal:

$$|\mathbf{F}_{13}| = |\mathbf{F}_{31}| = |\mathbf{F}_{12}| = |\mathbf{F}_{21}| = |\mathbf{F}_{23}| = |\mathbf{F}_{32}| = K\frac{q^2}{d^2}$$

But note that directions of the forces are not always the same.

Where can we add a fourth charge q_4 so that all charges are at rest? By symmetry, we expect the location of a positive fourth charge to be at the center of the triangle. But we need to determine the amount of charge needed to keep the triangle from expanding (forces must balance, so that charges are static).



For particle 1: $\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} = 0$
(condition for forces to balance at equilibrium)

Note that this is a vector equation, and the forces must separately balance in the *x* and *y* directions.
In component notation, these forces are:

$$\mathbf{F}_{12} = -F_{12}\cos 60^\circ\,\hat{\mathbf{x}} - F_{12}\sin 60^\circ\,\hat{\mathbf{y}}$$
$$\mathbf{F}_{13} = -F_{13}\hat{\mathbf{x}}$$
$$\mathbf{F}_{14} = F_{14}\cos 30^\circ\,\hat{\mathbf{x}} + F_{14}\sin 30^\circ\,\hat{\mathbf{y}}$$

So balancing each component of the forces gives:

$$\hat{\mathbf{x}}: \quad -F_{12}\cos 60^\circ - F_{13} + F_{14}\cos 30^\circ = 0$$
$$\hat{\mathbf{y}}: \quad -F_{12}\sin 60^\circ + F_{14}\sin 30^\circ = 0$$

Now $|\mathbf{F}_{12}| = |\mathbf{F}_{13}| = K\dfrac{q^2}{d^2}$

While $|\mathbf{F}_{14}| = K\dfrac{qq_4}{\ell^2}$ where $\ell = \dfrac{d}{\sqrt{3}}$

Let's take the equation balancing forces in $\hat{\mathbf{y}}$:

$$-K\frac{q^2}{d^2}\cdot\frac{\sqrt{3}}{2}+K\frac{qq_4}{\left(d^2/3\right)}\cdot\frac{1}{2}=0$$

$$-q\sqrt{3}+3q_4=0$$

$$\Rightarrow q_4=\frac{q}{\sqrt{3}}>0 \qquad \text{whereas } q_1=-q$$

This solves for the fourth charge, but it is useful to cross-check with the $\hat{\mathbf{x}}$ force equation to check for any mistakes:

$$-K\frac{q^2}{d^2}\cdot\frac{1}{2}-K\frac{q^2}{d^2}+K\frac{qq_4}{\left(d^2/3\right)}\cdot\frac{\sqrt{3}}{2}=0$$

$$-\frac{3}{2}q+3q_4\frac{\sqrt{3}}{2}=0$$

$$\Rightarrow q_4=\frac{q}{\sqrt{3}}$$

which checks out. We could also balance the forces acting on any of the other particles as well to determine the fourth charge.