Electric Potential

Disclaimer: These lecture notes are not meant to replace the course textbook. The content may be incomplete. Some topics may be unclear. These notes are only meant to be a study aid and a supplement to your own notes. Please report any inaccuracies to the professor.

Work and Potential Energy

Applying a force over a distance requires work:

$W = \mathbf{F} \cdot \mathbf{d}$ if $\mathbf{F}$ and $\mathbf{d}$ are constant

$W_{12} = \int_i^f \mathbf{F} \cdot d\mathbf{s}$ otherwise to move the object from initial point i to final point f

The work done by a force on an object to move it from point *i* to point *f* is opposite to the change in the potential energy:

$$W = -\Delta U = -\left(U_f - U_i\right)$$

In other words, if the work expended by the force is positive, the potential energy of the object is lowered. For example, if an apple is dropped from the branch of a tree, the force of gravity does work to move (accelerate actually) the apple from the branch to the ground. The apple now has less gravitational potential energy.

These concepts are independent of the type of force. So the same principal also applies to the electric field acting on an electric charge.

We define the <u>electric potential</u> as the potential energy of a positive test charge divided by the charge q_0 of the test charge.

$$V = \frac{U}{q_0}$$

It is by definition a scalar quantity, not a vector like the electric field.

The SI unit of electric potential is the <u>Volt</u> (V) which is 1 Joule/Coulomb. The units of the electric field, which are *N/C*, can also be written as *V/m* (discussed later).

Changes in the electric potential similarly relate to changes in the potential energy:

$$\Delta V = \frac{\Delta U}{q_0}$$

So we can compute the change in potential energy of an object with charge q crossing an electric potential difference:

$$\Delta U = q \Delta V$$

This motivates another unit for potential energy, since often we are interested in the potential energy of a particle like the electron crossing an electric potential difference. Consider an electron crossing a potential difference of 1 volt:

$$\Delta U = q \Delta V = e \Delta V = \left(1.6 \times 10^{-19} \text{ C}\right)\left(1 \text{ V}\right) = 1.6 \times 10^{-19} \text{ J} = 1 \text{ eV}$$

This is a tiny number, which we can define as one <u>electron-volt</u> (abbreviated "eV"). It is a basic unit used to measure the tiny energies of subatomic particles like the electron. You can easily convert back to the SI unit Joules by just multiplying by the charge of the electron, e.

A common convention is to set the electric potential at infinity (i.e. infinitely far away from any electric charges) to be zero. Then the electric potential at some point r just refers to the change in electric potential in moving the charge from infinity to point r.

$$\Delta V = V_r - V_\infty \rightarrow V_r$$

The work done by the electric field in moving an electric charge from infinity to point r is given by:

$$W = -\Delta U = -q \Delta V = -q\left(V_r - V_\infty\right) = -q V_r$$

where the last step is done by our convention. But keep in mind that it is only the differences in electric potential that have any meaning. A constant offset in electric potential or potential energy does not affect anything.

Electric Potential from Electric Field

Consider the work done by the electric field in moving a charge q_0 a distance ds:

$$dW = \mathbf{F} \cdot d\mathbf{s} = q_0 \mathbf{E} \cdot d\mathbf{s}$$

The total work done by the field in moving the charge a macroscopic distance from initial point i to final point f is given by a line integral along the path:

$$W = q_0 \int_i^f \mathbf{E} \cdot d\mathbf{s}$$

This work is related to the negative change in potential energy or electric potential:

$$\frac{W}{q_0} = -\Delta V = -\left(V_f - V_i\right)$$

$$\Rightarrow \Delta V = V_f - V_i = -\int_i^f \mathbf{E} \cdot d\mathbf{s} = \int_f^i \mathbf{E} \cdot d\mathbf{s}$$

The last step changes the direction of the integration and reverses the sign of the integral.

Equipotential Surfaces

Equipotential surfaces are surfaces (not necessarily physical surfaces) which are at equal electric potential. Thus, between any 2 points on the surface $\Delta V = 0$. This implies that no work can be done by the electric field to move an object along the surface, and thus we must have $\mathbf{E} \cdot d\mathbf{s} = 0$

Therefore, equipotential surfaces are always perpendicular to the direction of the electric field (the field lines).
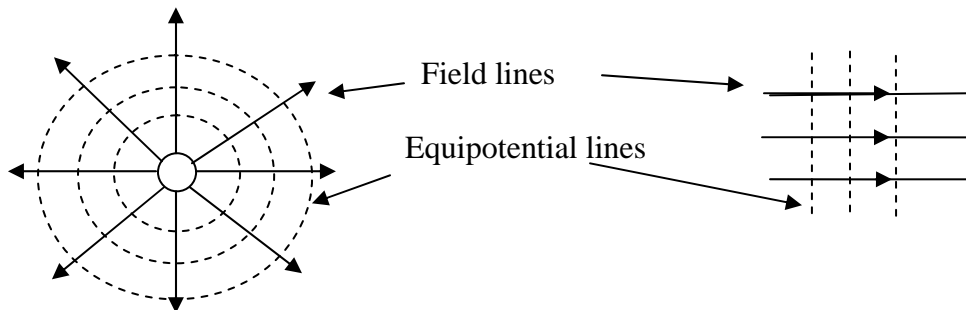
The potential lines indicate surfaces at the same electric potential, and the spacing is a measure of the rate of charge of the potential. The lines themselves have no physical meaning.

Potential of a Point Charge

Let's calculate the electric potential at a point a distance r away from a positive charge q. That is, let us calculate the electric potential difference when moving a test charge from infinity to a point a distance r away from the primary charge q.

$$\Delta V = V_r - V_\infty = -\int_\infty^r \mathbf{E} \cdot d\mathbf{s}$$

Let us choose a radial path. Then $\mathbf{E} \cdot d\mathbf{s} = -|\mathbf{E}||ds|$ since the field points in the opposite direction of the path. However, if we choose integrating variable dr, then $ds = -dr$ since r points radially outward like the field. We thus have:

$$\Delta V = -\int_{\infty}^{r} E \cdot ds = -\int_{\infty}^{r} E dr'$$

$$= -\int_{\infty}^{r} K \frac{q dr'}{r'^2} = -Kq \int_{\infty}^{r} \frac{dr'}{r'^2} = Kq \frac{1}{r'}\Big|_{\infty}^{r} = K \frac{q}{r}$$

Since the electric potential is chosen (and shown here) to be zero at infinity, we can just write for the electric potential a distance r away from a point charge q:

$$V(r) = K \frac{q}{r}$$

It looks similar to the expression for the magnitude of the electric field, except that it falls off as $1/r$ rather than $1/r^2$.

We also could integrated in the opposite sense:

$$\Delta V = V_{\infty} - V_{r} = -\int_{r}^{\infty} \mathbf{E} \cdot d\mathbf{s}$$

Then $\mathbf{E} \cdot d\mathbf{s} = |\mathbf{E}| dr$

$$\Delta V = -V_{r} = -\int_{r}^{\infty} \mathbf{E} \cdot d\mathbf{s} = -\int_{r}^{\infty} E dr'$$

$$= -\int_{r}^{\infty} K \frac{q dr'}{r'^2} = -Kq \int_{r}^{\infty} \frac{dr'}{r'^2} = Kq \frac{1}{r'}\Big|_{r}^{\infty} = -K \frac{q}{r}$$

$$V_{r} = K \frac{q}{r}$$

Potential of Many Point Charges

By the superposition principal, the electric potential arising from many point charges is just:

$$V = \sum_{i} K \frac{q_i}{r_i}$$

where q_i is the charge of the ith charge, and r_i is the distance from the charge to some point P where we wish to know the total electric potential. The advantage of this calculation is that you only have to linearly add the electric potential arising from each point charge, rather than adding each vector component separately as in the case of the electric field.

Electric Dipole

Let's see how to calculate the electric potential
at point *P* due to an electric dipole.

By the superposition principle, the total potential is:

$$V = V_+ - V_- = K\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

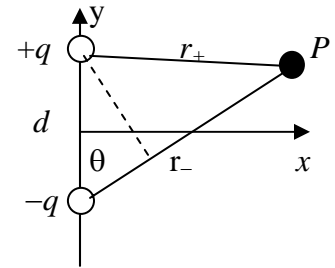$$V = Kq\left(\frac{r_- - r_+}{r_+ r_-}\right)$$

where r_+ is the distance from the positive charge to point P, and r_- the distance from the
negative charge.

Now for large distances, $r \gg d \Rightarrow r_- - r_+ \approx d \cos\theta$, where d is the separation of the
electric dipole.

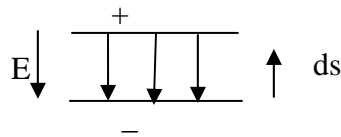$$V = Kq\frac{d\cos\theta}{r^2} = K\frac{p\cos\theta}{r^2}$$

where $r = r_+ \approx r_-$ and $p \equiv qd$ is the electric dipole moment.

Potential of Continuous Charge Distributions

Potential between 2 Parallel Plates

Let's calculate the electric potential difference between 2 large parallel conducting plates
separated by a distance *d*, with the upper plate (denoted "+") at higher electric potential
than the lower.

From what we learned by Gauss's Law and conductors, we know that the electric field

arising from a conductor with a charge density σ is $|\mathbf{E}| = \dfrac{\sigma}{\varepsilon_0}$. It is thus a constant between

the two plates in this example. The electric potential difference is given by a line integral:

$$\Delta V = V_+ - V_- = -\int_-^+ \mathbf{E} \cdot d\mathbf{s} \qquad \mathbf{E} \cdot d\mathbf{s} = -|\mathbf{E}| ds \quad \text{opposite directions}$$
$$\Rightarrow \Delta V = |\mathbf{E}| d$$

Another way to view this result is that if we apply an electric potential difference between two conducting plates (large compared to their separation d), the magnitude of the electric field between them is:

$$|\mathbf{E}| = \frac{\Delta V}{d}$$

This motivates the alternate units for electric field of *V/m*.

See more examples in the textbook on continuous charge distributions!

Electric Field from Electric Potential

We have seen in the previous example of the electric potential between two parallel plates, that

$$|\mathbf{E}| = \frac{\Delta V}{\Delta s}$$

where Δs is the spacing between the plates, where the path is parallel to the field direction (and perpendicular to equipotential surfaces). In fact, the field points in direction opposite to increasing electric potential difference along path **s**:

$$\mathbf{E} = -\frac{\Delta V}{\Delta s} \hat{\mathbf{s}}$$

Now in the infinitesimal limit,

$$\mathbf{E} = -\frac{dV}{ds} \hat{\mathbf{s}}$$

which applies to the field calculated in any region, uniform or not. Writing this into the usual Cartesian coordinates:

$\mathbf{E} = -\nabla V$ where ∇ is the <u>gradient operator</u>. It is a short-hand for:

$$E_x = -\frac{\partial V}{\partial x}$$

$$E_y = -\frac{\partial V}{\partial y}$$

$$E_z = -\frac{\partial V}{\partial z}$$

So the electric field is related to the negative rate of change of the electric potential.

This is a specific manifestation of a more general relation that a force is related to the rate of change of the corresponding potential energy:

$$\mathbf{F} = -\nabla U \qquad (\text{in one dimension: } F = -\frac{dU}{dx})$$

For the case of the electric field, $\mathbf{F} = q\mathbf{E}$ and $U = qV$, so

$$q\mathbf{E} = -q\nabla V \Rightarrow \mathbf{E} = -\nabla V$$

Conductors and Electric Potential

Recall that the valence electrons in a conductor are free to move, but that in electrostatic equilibrium they have no net velocity. Another consequence of this is that:

$\Delta V = 0$ across a conductor

If not, electrons would move from higher to lower potential, and thus not be in static equilibrium. This implies that the surface of the conductor, no matter what shape, is also an equipotential surface. We learned already that the electric field is perpendicular to the surface of a conductor (otherwise charges would accelerate along the surface), and equipotential lines are always perpendicular to the electric field lines.

Equipotential line

Surface of conductor
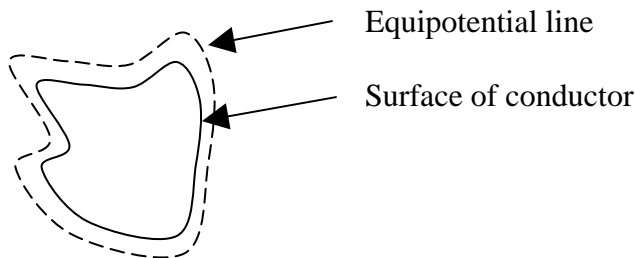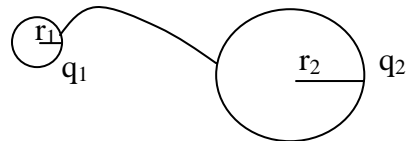
Example:

Let's consider as an example 2 conducting spheres connected by a thin conducting wire. One sphere has a smaller radius (r_1) than the other (r_2); and the charges on the two spheres are q_1 and q_2 respectively.

By the above argument, all surfaces are at the same electric potential. Let's raise the entire system to potential V with respect to a point infinitely far away. The two spheres must have the same potential, so by equating the potential energy of each charged sphere (which is the same as that of a point charge at the center of the sphere) we get:

$$K \frac{q_1}{r_1} = K \frac{q_2}{r_2}$$

$$\Rightarrow \frac{q_1}{q_2} = \frac{r_1}{r_2}$$

Now let's determine the surface charge densities. Since $\sigma = \dfrac{q}{4\pi r^2}$, the last equation can be written:

$$\frac{\sigma_1 4\pi r_1^2}{\sigma_2 4\pi r_2^2} = \frac{r_1}{r_2}$$

$$\Rightarrow \frac{\sigma_1}{\sigma_2} = \frac{r_2}{r_1}$$

i.e. the surface charge density is inversely proportional to the radius of the sphere.

Now the magnitude electric field at the surface of the sphere is:

$$|\mathbf{E}| = K\frac{q}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma 4\pi r^2}{r^2} = \frac{\sigma}{\varepsilon_0}$$

Thus, the field strength is proportional to the surface charge density, which is inversely proportional to the radius of the sphere.

For a large enough charge q_1 and small enough radius r_1, the breakdown electric field strength in air could be exceeded (3×10^6 V/m) and a discharge (lightening bolt) occur.

This is the basis of a lightening rod. Let the large sphere represent a large surface, such as the Earth, and the small sphere a small narrow point such as a rod. If the two surfaces accrue a large charge, such as during a lightening storm, the electric field is strongest at the narrow rod and a breakdown is most likely to occur there. Do not stand near tall narrow objects (like trees!) in an electrical storm!