

Maxwell's Equations and EM Waves

Disclaimer: These lecture notes are not meant to replace the course textbook. The content may be incomplete. Some topics may be unclear. These notes are only meant to be a study aid and a supplement to your own notes. Please report any inaccuracies to the professor.

Maxwell's Equations

Let's summarize all the electromagnetic equations we have learned so far, both integral and differential forms:

1. Gauss' Law for electric field:

$$\Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{enc}}}{\epsilon_0} \quad \text{or} \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

2. Gauss' Law for magnetic field:

$$\Phi_B = \oint_S \mathbf{B} \cdot d\mathbf{A} = 0 \quad \text{or} \quad \nabla \cdot \mathbf{B} = 0 \quad (\text{no magnetic monopole charges})$$

3. Faraday's Law of Induction:

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{A} \quad \text{or} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

4. Ampere's Law and Maxwell's Law of Induction:

$$\oint_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 \epsilon_0 \frac{d}{dt} \int_S \mathbf{E} \cdot d\mathbf{A} + \mu_0 i_{\text{enc}} \quad \text{or} \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}$$

Derivation of Electromagnetic Wave Equation

Now let's see how we can combine the differential forms of Maxwell's equations to derive a set of differential equations (wave equations) for the electric and magnetic fields. Let's assume we solve these equations in a region without any electric charges present ($\rho=0$) or any currents ($j=0$).

Start with Maxwell's Law:

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Now take the curl of this equation:

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{E})$$

Now it can be shown as a proof in vector calculus that:

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$$

where $\nabla \cdot \mathbf{B} = 0$ by Gauss' Law for magnetic fields. And since Faraday's Law tells us:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

then we get:

$$\nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

which is a second order differential equation for each of the 3 components of the magnetic field. (It is a wave equation it turns out).

Now we can follow a similar derivation for the electric field starting with Faraday's Law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Now take the curl of this equation:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B})$$

Now using the same vector calculus proof:

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

where $\nabla \cdot \mathbf{E} = 0$ by Gauss' Law for electric fields in vacuum. And since Maxwell's Law tells us:

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

then we get:

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Interestingly this is the exact same differential wave equation as for magnetic fields!

Solution to Wave Equation

The general form of the wave equation is:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{F} = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \mathbf{F}$$

where we have expanded the Laplacian (∇^2) operator and defined

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c = 3 \times 10^8 \text{ m/s.}$$

Solutions to this partial differential equation have the general form:

$$\mathbf{F}(\mathbf{x}, t) = \mathbf{F}_m \sin(\mathbf{k} \cdot \mathbf{x} - \omega t - \phi)$$

But let's use complex notation to see how the solution works:

$$\mathbf{F}(\mathbf{x}, t) = \mathbf{F}_m \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$$

where we just take the imaginary or real part to get the physical solution.

Plugging into the wave equation yields:

$$i^2 (k_x^2 + k_y^2 + k_z^2) \mathbf{F}(\mathbf{x}, t) = \frac{1}{v^2} (-i\omega)^2 \mathbf{F}(\mathbf{x}, t)$$

$$\Rightarrow k^2 = \frac{\omega^2}{v^2}$$

$$v = \frac{\omega}{k}$$

Thus, the solution to the wave equation that is a consequence of Maxwell's equations in vacuum is a sinusoidally varying function for both the electric and magnetic fields. It is a traveling wave solution, which becomes more apparent if we write the solution in this form:

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_m \exp[i\mathbf{k} \cdot (\mathbf{x} - \mathbf{v}t)] \quad (\text{or } \mathbf{E}_m \sin[\mathbf{k} \cdot (\mathbf{x} - \mathbf{v}t)])$$

$$\mathbf{B}(\mathbf{x}, t) = \mathbf{B}_m \exp[i\mathbf{k} \cdot (\mathbf{x} - \mathbf{v}t)] \quad (\text{or } \mathbf{B}_m \sin[\mathbf{k} \cdot (\mathbf{x} - \mathbf{v}t)])$$

Here:

$\mathbf{E}_m, \mathbf{B}_m =$ amplitude of electric and magnetic fields

$\omega =$ angular frequency (rad/s) $= 2\pi / f$

$f =$ cyclic frequency (s^{-1} , Hz)

$T =$ period (s)

$k =$ wavenumber (m^{-1}) $= 2\pi / \lambda$

$\lambda =$ wavelength (m)

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = c \quad \text{phase velocity of wave (m/s)}$$

Note that Maxwell's Equations predict a unique velocity for the electromagnetic waves, which is just c , the speed of light. Thus, for Maxwell's equations to be correct in all reference frames we are led to Einstein's theory of Special Relativity!

The wavenumber \mathbf{k} is actually a vector, as is the velocity \mathbf{v} . They both point in the direction of the traveling wave. Notice that as time increases, $\mathbf{k} \cdot \mathbf{x}$ must increase proportionately to maintain the same phase.

Now both the electric and magnetic fields have the same form of the wave solution. These solutions have field directions \mathbf{E}_m and \mathbf{B}_m , and the solution itself fills all space (a little unrealistic!) But the magnetic and electric fields are also related. Let's explore some properties of the derived electromagnetic wave:

Consider Gauss' Law:

$$\begin{aligned} \nabla \cdot \mathbf{E} = 0 &\Rightarrow \nabla \cdot (\mathbf{E}_m \exp(i\mathbf{k} \cdot \mathbf{x} - \omega t)) = 0 \\ &\Rightarrow \mathbf{k} \cdot \mathbf{E}_m = 0 \end{aligned}$$

Which implies that the electric field direction is perpendicular to the velocity direction, or wave direction. We can derive the same thing for magnetic field direction:

$$\begin{aligned} \nabla \cdot \mathbf{B} = 0 &\Rightarrow \nabla \cdot (\mathbf{B}_m \exp(i\mathbf{k} \cdot \mathbf{x} - \omega t)) = 0 \\ &\Rightarrow \mathbf{k} \cdot \mathbf{B}_m = 0 \end{aligned}$$

Now consider Faraday's Law:

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ &\Rightarrow i\mathbf{k} \times \mathbf{E}_m \exp(i\mathbf{k} \cdot \mathbf{x} - \omega t) = i\omega \mathbf{B}_m \exp(i\mathbf{k} \cdot \mathbf{x} - \omega t) \\ &\Rightarrow \mathbf{k} \times \mathbf{E}_m = \omega \mathbf{B}_m \end{aligned}$$

Note that the last step can only be satisfied if the electric and magnetic waves have exactly the same time and space form (same phase, velocity, wavenumber).

Now both \mathbf{E}_m and \mathbf{B}_m are perpendicular to \mathbf{k} , and by this equation \mathbf{B}_m is perpendicular to \mathbf{E}_m as well!

$$\Rightarrow \mathbf{k} \perp \mathbf{E}_m \perp \mathbf{B}_m$$

This is an important feature of electromagnetic waves. They are transversely polarized, and the electric and magnetic components are perpendicular to each other.

Moreover, the magnitudes are related:

$$\boxed{\frac{E_m}{B_m} = \frac{\omega}{k} = c}$$

Thus, there is only one independent wave solution. If you know the form of the electric field, the magnetic field is completely specified by the above relations.

Note that there is another way to represent the above equation if we substitute in for the angular frequency and the wavenumber:

$$\frac{\omega}{k} = \frac{2\pi f}{(2\pi/\lambda)}$$

$$\Rightarrow \boxed{f\lambda = c} \quad (\text{or } v \text{ more generally})$$

Now the wide range of electromagnetic wave phenomena are described by just different choices of the frequency (or alternatively wavelength, as it is not independent). Lowest in frequency (largest wavelength) are radio waves (kHz-MHz), increasing to microwaves (GHz), infrared radiation (10^{14} Hz), visible light ($\sim 10^{15}$ Hz), ultraviolet light, x-rays, and finally gamma rays!