



Relativity 2

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Einstein's Postulates

The absence of any fringe shift in the Michelson-Morely experiment for any orientation of the interferometer and for any time of the year negated the ether hypothesis for light propagation. Light waves are oscillations of the electromagnetic field, and no propagation medium is necessary, unlike sound waves. However, if Galilean transformations are correct, then Maxwell's equations must be modified for every possible reference frame to account for different velocities for the speed of light. Einstein assumed the opposite: that Maxwell's equations are fundamentally correct, but that our intuitive Galilean transformation is not. This led to the following two postulates:

1. The laws of physics, including electromagnetism, are the same in all inertial frames.
2. Every observer measures the same value c for the speed of light (in vacuum) in all inertial frames.

The second postulate is really a consequence of the first, because if Maxwell's equations hold in all inertial frames, then the only possible value for the speed of light is c . These postulates embody Einstein's **Special Theory of Relativity**, first published in 1905 in a paper titled On the Electrodynamics of Moving Bodies. Later he would incorporate gravity and acceleration in his General Theory of Relativity. As in Newtonian Relativity, there is no way to detect absolute motion. Only the relative velocities between two inertial reference frames matters.

These seemingly simple postulates have extraordinary consequences. For example, when you turn on the headlights of a car, the light beam leaves the car at a relative velocity of $c = 3.0 \times 10^8$ m/s. However, someone standing on the sidewalk also measures the speed of the light beam as c independent of the velocity of the car! How can this be? As we shall see, our concepts of space and time must be modified.

Basic Definitions

Events are physical phenomena that occur independent of any reference frame. For example: a flash, explosion, return of a spaceship, or disintegration of a subatomic particle.

Observers record events, both the time and spatial coordinates, in a particular reference frame. For example, Mission Control in Houston marking down the time and location of the splashdown of a space capsule. The reference frame in this case is the Earth.

Simultaneous events occur when the light signals from two events reach an observer at the same time

Relativity of Simultaneity:

Two events simultaneous in one inertial frame are not simultaneous in any other frame. This is a consequence of Einstein's Postulates.

Proper time is the time difference between two events occurring at the same position (Denoted by t_0 or τ).

Rest frame is the inertial frame where two events are only separated by time. The frame in which the proper time is measured

Proper length is the distance between two positions at rest, the length measured in the rest frame. (Denoted by L_0).

Now that we are armed with these definitions, let's explore the consequences of the constancy of the speed of light in all inertial frames.

Time Dilation

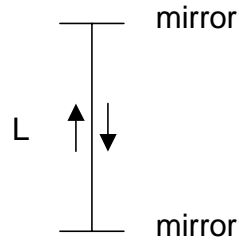
We explore the rate of time in different inertial frames by considering a special kind of clock – a light clock – which is just one arm of an interferometer. Consider a light pulse bouncing vertically between two mirrors. We analyze the time it takes for the light pulse to complete a round trip both in the rest frame of the clock (labeled S'), and in an inertial frame where the clock is observed to move horizontally at a velocity v (labeled S).

In the rest frame S'

$$t_1' = \frac{L}{c} = \text{time up}$$

$$t_2' = \frac{L}{c} = \text{time down}$$

$$\tau = t_1' + t_2' = \frac{2L}{c}$$



Now put the light clock on a spaceship, but measure the roundtrip time of the light pulse from the Earth frame S :

$$t_1 = \frac{t}{2} = \text{time up}$$

$$t_2 = \frac{t}{2} = \text{time down}$$

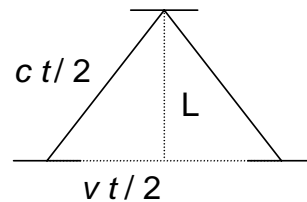
The speed of light is still c in this frame, so

$$L^2 + v^2 t^2 / 4 = c^2 t^2 / 4$$

$$L^2 = (c^2 - v^2) t^2 / 4$$

$$t^2 = \frac{4L^2}{c^2 - v^2}$$

$$t = \frac{2L}{c} \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{\tau}{\sqrt{1 - v^2/c^2}}$$



So the time it takes the light pulse to make a roundtrip in the clock when it is moving by us appears longer than when it is at rest. We say that time is **dilated**. It also doesn't matter which frame is the Earth and which is the clock. Any object that moves by with a significant velocity appears to have a clock running slow. We summarize this effect in the following relation:

$$t = \gamma \tau \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \geq 1$$

Length Contraction

Now consider using a light clock to measure the length of an interferometer arm. In particular, let's measure the length along the direction of motion.

In the rest frame S'

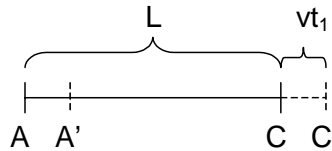
$$L_0 = \frac{c\tau}{2}$$

Now put the light clock on a spaceship, but measure the roundtrip time of the light pulse from the Earth frame S :

$t_1 =$ time out

$t_2 =$ time back

$$t = t_1 + t_2$$



$$L + vt_1 = ct_1 \Rightarrow t_1 = \frac{L}{c - v}$$

$$L - vt_2 = ct_2 \Rightarrow t_2 = \frac{L}{c + v}$$

$$t = t_1 + t_2 = \frac{2Lc}{c^2 - v^2} = \frac{2L}{c} \frac{1}{1 - v^2/c^2}$$

$$L = \frac{ct}{2} (1 - v^2/c^2)$$

But, $t = \frac{\tau}{\sqrt{1 - v^2/c^2}}$ from time dilation

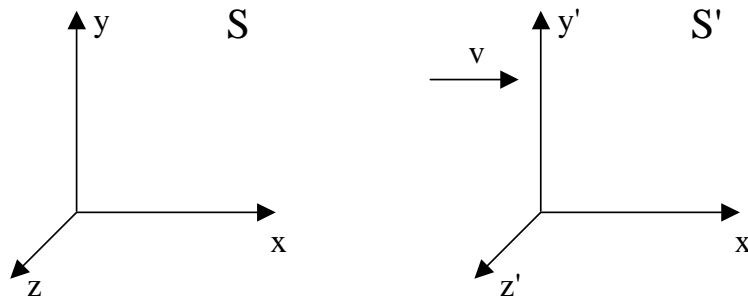
$$\boxed{L = \frac{L_0}{\gamma} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \geq 1}$$

In other words, the length of the interferometer arm appears contracted when it moves by us. This is known as the **Lorentz-Fitzgerald contraction**. It is closely related to time dilation. In fact, one implies the other, since we used time dilation to derive length contraction.

Time dilation and length contraction are consequences of the assumption that all observers measure the same value for the speed of light. This means that time runs at different rates for different inertial frames. There is no absolute time, time only has a relative meaning. Likewise, length also has only a relative meaning. Everything depends on the relative velocity between two objects. We only notice these strange effects when the velocity is near c , however.

The Lorentz Transformation

We are now ready to derive the correct transformation equations between two inertial frames in Special Relativity, which modify the Galilean Transformation. We consider two inertial frames S and S' , which have a relative velocity v between them along the x -axis.



Now suppose that there is a single flash at the origin of S and S' at time $t = t' = 0$, when the two inertial frames happen to coincide. The outgoing light wave will be spherical in shape moving outward with a velocity c in *both* S and S' by Einstein's Second Postulate.

$$x^2 + y^2 + z^2 = c^2 t^2$$

$$x'^2 + y'^2 + z'^2 = c^2 t'^2$$

We expect that the orthogonal coordinates will not be affected by the horizontal velocity:

$$y' = y$$

$$z' = z$$

But the x coordinates will be affected. We assume it will be a linear transformation:

$$x' = k(x - vt)$$

$$x = k'(x' + vt')$$

But in *Relativity* the transformation equations should have the same form (the laws of physics must be the same). Only the relative velocity matters. So $k' = k$.

Consider the outgoing light wave along the x -axis ($y = z = 0$).

$$x' = ct' \quad \text{in frame } S'$$

$$x = ct \quad \text{in frame } S$$

Now plug these into the transformation equations:

$$x' = k(x - vt) = k(ct - vt) = kct(1 - v/c) \quad \text{and}$$

$$x = k(x' + vt') = k(ct' + vt') = kct'(1 + v/c)$$

Plug these two equations into the light wave equation:

$$ct' = x' = kct(1 - v/c)$$

$$ct = x = kct'(1 + v/c)$$

$$t' = kt(1 - v/c)$$

$$t = kt'(1 + v/c)$$

Plug t' into the equation for t :

$$t = k^2 t'(1 + v/c)(1 - v/c)$$

$$1 = k^2 (1 - v^2/c^2)$$

$$\Rightarrow k = \frac{1}{\sqrt{1 - v^2/c^2}} \equiv \gamma$$

So the modified transformation equations for the spatial coordinates are:

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

Now what about time?

$$x' = \gamma(x - vt)$$

$$x = \gamma(x' + vt') \quad \text{inverse transformation}$$

Plug one into the other:

$$x = \gamma(\gamma x - \gamma vt + vt')$$

Solve for t' :

$$x - \gamma^2 x + \gamma^2 vt = \gamma vt'$$

$$x(1 - \gamma^2) + \gamma^2 vt = \gamma vt'$$

$$x \frac{1 - v^2/c^2 - 1}{1 - v^2/c^2} + \gamma^2 vt = \gamma vt'$$

$$-\gamma^2 xv^2/c^2 + \gamma^2 vt = \gamma vt'$$

$$\Rightarrow t' = \frac{1}{\gamma v} (-\gamma^2 xv^2/c^2 + \gamma^2 vt)$$

$$t' = \gamma(t - vx/c^2)$$

So the correct transformation (and inverse transformation) equations are:

$x' = \gamma(x - vt)$	$x = \gamma(x' + vt')$
$y' = y$	$y = y'$
$z' = z$	$z = z'$
$t' = \gamma(t - vx/c^2)$	$t = \gamma(t' + vx'/c^2)$

The Lorentz
Transformation

Note the following features:

- We recover the Galilean transformation if $c \rightarrow \infty$ (or $v \rightarrow 0$) so that $\gamma \rightarrow 1$
- Space and time coordinates are mixed (x, t)
- No change in form of equations from one frame to another (Einstein's 1st postulate)
- Only relative velocities matter

Also, note that you can derive time dilation and length contraction from these equations. For example, if a clock sits at rest in frame S' at position $x'=0$, then an observer in frame S measures the period of the clock to be $T = \gamma \tau$.

Moreover, note that two events which are simultaneous in frame S' (say at time $t'=0$ and at positions x'_1 and x'_2) are not simultaneous in frame S ($t_1 \neq t_2$).