Relativity 4

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Relativistic Momentum

Newton's 2nd Law can be written in the form

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

where the non-relativistic momentum of a body is $\mathbf{p} = m\mathbf{u}$ where $\mathbf{u} = \frac{d\mathbf{x}}{dt}$. However, because of the Lorentz transformation equations, $\frac{d\mathbf{x}}{dt}$ is measured differently in different inertial frames. Thus, Newton's 2nd Law would not have the same form in different frames. We need a new definition of momentum to retain the definition of force as a change in momentum.

Suppose $\mathbf{p} = m\frac{d\mathbf{x}}{d\tau}$, where τ is the proper time in the object's rest frame. Every observer will agree on which frame is the rest frame. Also, since $y' = y$ and $z' = z$, the transverse momentum (p_y and p_z) will be invariant for a Lorentz transformation along the x axis. (This would not be the case if we did not use the proper time in the definition). We can rewrite this momentum definition as follows:

$$\mathbf{p} = m\frac{d\mathbf{x}}{d\tau} = m\frac{d\mathbf{x}}{dt}\frac{dt}{d\tau}$$

$$t = \gamma\tau \quad \Rightarrow \quad \frac{dt}{d\tau} = \gamma \qquad \text{From time dilation}$$

$$\boxed{\mathbf{p} = \gamma_u\, m\, \mathbf{u} \qquad \gamma_u = \frac{1}{\sqrt{1-u^2/c^2}}}$$

Recall that momentum is a vector quantity. Conservation of momentum, which still applies in Special Relativity, implies that each component of momentum is conserved.

Note that **u** is the velocity of the object in a reference frame, not the velocity of a reference frame relative to another.

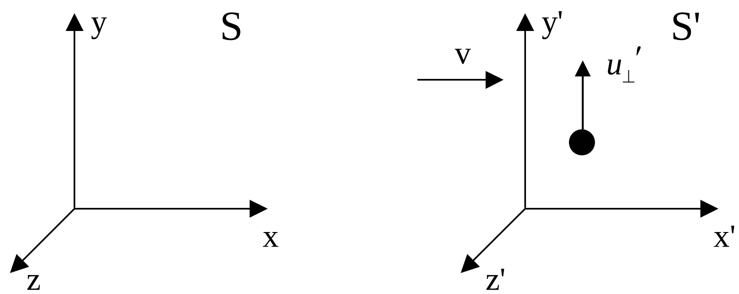
In this definition of momentum, the mass $m=m_0$ is the "rest mass". That is, it is the mass of an object in its rest frame. Sometimes γm is referred to as the "relativistic mass", such that we can retain the Newtonian definition of momentum as $\mathbf{p} = m\mathbf{u}$. In this sense, the mass of an object grows as its velocity increases. But this convenient trick can be problematic. As we shall see, the kinetic energy, for example, is not ½ mv^2.

Relativistic Force

With the previous relativistic definition for momentum, we can retain the usual definition for **force**:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}\left(m\frac{d\mathbf{x}}{d\tau}\right) = \frac{d}{dt}\left(\gamma_u \ m\mathbf{u}\right) \qquad \text{where } \mathbf{u} = \frac{d\mathbf{x}}{dt} \quad \text{and} \quad \gamma_u = \frac{1}{\sqrt{1-u^2/c^2}}$$

It is useful to consider how force transforms under a Lorentz Transformation:



According to the addition of velocity formulae, the transformation of the velocity perpendicular to the direction of the Lorentz Transformation is:

$$\mathbf{u}_\perp = \frac{\mathbf{u}_\perp{}'}{\gamma_v\left(1+vu_x{}'/c^2\right)} \quad \text{where} \quad \gamma_v = \frac{1}{\sqrt{1-v^2/c^2}}$$

So for the perpendicular force, which can be written as:

$$\mathbf{F}_\perp = m\frac{d}{dt}\frac{d\mathbf{x}}{d\tau} = m\frac{d}{d\tau}\mathbf{u}_\perp$$

it transforms as:

$$\mathbf{F}_\perp = m\frac{d}{d\tau}\frac{\mathbf{u}_\perp{}'}{\gamma_v\left(1+vu_x{}'/c^2\right)} = \frac{1}{\gamma_v\left(1+vu_x{}'/c^2\right)} m\frac{d\mathbf{u}_\perp{}'}{d\tau}$$

$$\mathbf{F}_\perp = \frac{\mathbf{F}_\perp'}{\gamma_v\left(1+vu_x{}'/c^2\right)}$$

where we assume no acceleration in the direction parallel to the transformation

Relativistic Energy

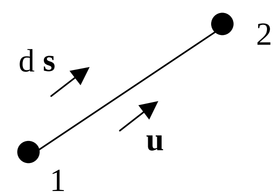
Now **work** is defined as force applied over a distance. It corresponds to the expended energy to accelerate a body. If the force and path are constant,

$$W = F \cdot d$$

More generally, if the force and path vary, then a line integral must be performed from initial position 1 to final position 2.

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{s}$$

The work applied to a body translates to a change in the kinetic energy since energy must be conserved. If we assume that the body is initially at rest, then the final kinetic energy is equal to the work expended:



$$W = K = \int \frac{d}{dt}(\gamma m\mathbf{u}) \cdot \mathbf{u}\, dt$$ where we have used $d\mathbf{s} = \mathbf{u}\, dt$

$$K = m\int dt \frac{d}{dt}(\gamma\mathbf{u}) \cdot \mathbf{u}$$

$$K = m\int_0^{\gamma U} u\, d(\gamma u)$$

Integrate by parts:

$$K = \gamma m U^2 - m\int_0^U \gamma u\, du$$

$$= \gamma m U^2 - m\int_0^U \frac{u\, du}{\sqrt{1-u^2/c^2}}$$

You can check this integral by differentiation

$$= \gamma m U^2 + mc^2\sqrt{1-u^2/c^2}\Big|_0^U$$

$$= \gamma m U^2 + mc^2\sqrt{1-U^2/c^2} - mc^2$$

$$= \gamma\left[mU^2 + mc^2\left(1-U^2/c^2\right)\right] - mc^2$$

Thus, we get for the relativistic kinetic energy:

$$\boxed{K = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2}$$

This final expression for the kinetic energy looks like nothing like the non-relativistic equation $K = \frac{1}{2}mu^2$. However, if we consider velocities much less than the speed of light, we can see the correspondence:

$\gamma = \left(1 - u^2 / c^2\right)^{-1/2} \approx 1 + \frac{1}{2} u^2 / c^2 + \cdots$ using the binomial expansion

$\Rightarrow K = (\gamma - 1)mc^2 \approx \frac{1}{2}\frac{u^2}{c^2} mc^2 = \frac{1}{2} mu^2$ for $u << c$

So at low velocities there is no difference between the definition of kinetic energy in Special Relativity from that in Newtonian Mechanics.

Now let's consider the opposite limit when the velocity approaches the speed of light. In that case, the kinetic energy becomes infinite as the relativistic factor γ goes to infinity. This is another way of saying that objects cannot exceed the speed of light, because it would take an infinite amount of energy.

Now let's rewrite the equation involving the kinetic energy:

$$E \equiv \gamma mc^2 = K + mc^2$$

This equation has the form of kinetic energy plus potential energy equals total energy. What is the potential energy? It is the term:

$$E_0 = mc^2$$

which we refer to as the **rest energy**. As you know, this is Einstein's famous equation that tells us that mass is another form of energy. Mass can be converted into energy and vice versa. How much energy? Let's see:

Example: Suppose that a 1 kg mass moves at a velocity u = 1 m/s. The kinetic energy is ½ $m u^2$ = ½ J. (We can use the non-relativistic equation because the velocity is much much smaller than the speed of light.) The rest mass energy is $mc^2 = 9.0 \times 10^{16}$ J. Clearly there is a tremendous amount of energy in 1 kg of mass. That is why nuclear weapons have the power that they do, because they convert a significant amount of mass into energy.

Conservation of Energy:

We have learned in earlier physics courses that kinetic energy does not have to be conserved in an inelastic collision. Likewise, mass does not have to be conserved since it can be converted into energy. However, the total energy (kinetic, rest mass, and all other potential energy forms) is always conserved in Special Relativity. Momentum and energy are conserved for both elastic and inelastic collisions when the relativistic definitions are used.

Relationship between Energy and Momentum

Using the **Newtonian** definitions of energy and momentum,
$E = \frac{1}{2}mu^2$ and $p = mu$, we can write:

$$E = \frac{p^2}{2m}$$

Now consider the **relativistic** definitions:

$$E = \gamma mc^2$$
$$p = \gamma mu$$
$$p^2 = \gamma^2 m^2 u^2$$
$$p^2c^2 = \gamma^2 m^2 u^2 c^2 = \gamma^2 m^2 c^4 \frac{u^2}{c^2} = \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2}\right)$$
$$p^2c^2 = \gamma^2 m^2 c^4 - m^2 c^4$$

But $E = \gamma mc^2$

So $p^2c^2 = E^2 - m^2c^4$

Thus the equivalent relationship between energy and momentum in Relativity is:

$$E^2 = p^2c^2 + m^2c^4 \quad \text{or equivalently} \quad m^2c^4 = E^2 - p^2c^2$$

This is another example of **Lorentz Invariance**. No matter what inertial frame is used to compute the energy and momentum, $E^2 - p^2c^2$ always given the rest energy of the object. Energy and momentum take the role of time and space in the other Lorentz invariant quantity Δs. In fact, we refer to (t, x, y, z) and (E, p_x, p_y, p_z) as **four-vectors**, and the "lengths" of these vectors are these Lorentz-invariant expressions we derived.

Particles without mass are a special case

$\Rightarrow E = pc$

E and *pc* can also be written: $E = \gamma mc^2$ and $pc = \gamma muc$.
The only way we can reconcile these last two definitions with $E = pc$ is to set the velocity to *c*. **Massless particles must travel at the speed of light.**
As we will learn, light itself is composed of particles (photons). To travel at the speed of light, these particles must be massless.

The Electron-Volt Energy Unit

The Lorentz force law is $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, where $\mathbf{E}$ is the electric field and $\mathbf{B}$ is the magnetic field. The work done to move a charged particle in an electric field only is:

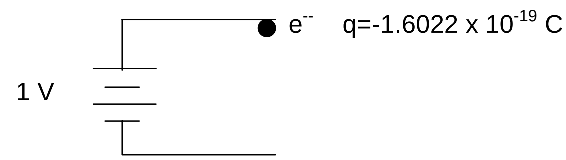
$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{s} = q\int_1^2 \mathbf{E} \cdot d\mathbf{s}$$
$$= q\left(V_2 - V_1\right)$$

The electric potential is ϕ (such that the electric field $\mathbf{E} = -\nabla V$). We can summarize the work done by;

$$\boxed{W = q\Delta V \qquad \Delta V = \text{potential difference}}$$

Consider the work done to move an electron across a potential difference of 1 Volt?

$$W = \left(-1.6022 \times 10^{-19}\ \text{C}\right)\left(-1\ \text{V}\right) = 1.6022 \times 10^{-19}\ \text{J}$$



This is a very small unit! We define it as a new unit of energy, the **electron-volt**:

$$\boxed{1\ \text{eV} = 1.6022 \times 10^{-19}\ \text{J}}$$

Example: Express the electron rest mass energy in this new unit:

$$E_0 = m_e c^2 = \left(9.11 \times 10^{-31}\ \text{kg}\right)\left(3.0 \times 10^{8}\ \text{m / s}\right)^2 \frac{1\ \text{eV}}{1.6022 \times 10^{-19}\ \text{J}}$$
$$E_0 = 511{,}000\ \text{eV} \quad (\text{or } 511\ \text{keV},\ 0.511\ \text{MeV},\ 0.000511\ \text{GeV})$$

We also can define new units for mass and momentum. For example, the mass of the electron can be expressed $m_e = 0.511\ \text{MeV} / c^2$. In other words, if you multiply the mass by c^2, you get the rest energy in electron-volts.

Similarly, we know that *pc* has units of energy, so momentum can be expressed in units like MeV / *c*. In other words, if you multiply by *c*, you get an energy in electron-volts.

Invariant Mass

We can now apply the relativistic definitions of energy and momentum to calculations of particle collisions. In particular, we can compute the rest mass of a particle formed when two particles annihilate into pure energy and then form a new particle.

Example: An electron and a positron (an anti-electron) annihilate with equal and opposite momentum: $p = 1.55 \text{ GeV}/c$. (Note the new momentum unit). The collision produces a new particle called the *J/ψ* in the following reaction: $e^- + e^+ \rightarrow J/\psi$. What is the mass of this new particle?

We need to compute the **invariant mass** of the electron-positron initial state to determine the rest mass of the new particle:

$$Mc^2 = \sqrt{E_{tot}^2 - p_{tot}^2 c^2} \quad \text{where } E_{tot} \text{ and } p_{tot} \text{ are the total energy and momentum}$$

$$p_{tot} = p_1 + p_2 = 1.55 \text{ GeV}/c - 1.55 \text{ GeV}/c = 0 \quad \text{by conservation of momentum}$$

$$E_{tot} = E_1 + E_2 \quad \text{by conservation of energy}$$

$$E_1 = \sqrt{p_1^2 c^2 + m^2 c^4} = \sqrt{(1.55 \text{ GeV})^2 + (0.000511 \text{ GeV})^2} \approx 1.55 \text{ GeV}$$

$$E_1 = E_2 \quad \text{because the magnitude of the momentum (and mass) is the same}$$

$$\Rightarrow Mc^2 = E_{tot} = 1.55 + 1.55 \text{ GeV} = 3.1 \text{ GeV}$$

The *J/ψ* particle has a mass of 3.1 GeV/*c*. Note that we have made extensive use of the new Lorentz-invariant quantity involving energy and momentum

Binding Energy

As we have learned, mass is a form of potential energy. It can be converted into energy, or energy can be converted into mass. Because of this, **mass does not have to be conserved** in reactions. If you throw two balls at each other and they stick together (an **inelastic** collision), the resulting mass is not necessarily the sum of the individual masses of the two balls.

This surprising result makes sense when we consider that mass is just another form of potential energy. When two balls stick together, there must be some attractive force holding the composite system together. In the case of the hydrogen atom, an electron and proton are bound by an attractive electromagnetic force. To separate the electron and proton (*i.e.* ionize hydrogen), one must overcome the attractive force, and that takes energy. In other words, the particles have larger electromagnetic potential energy when separated than together. This potential energy is:

$$V = \frac{-e^2}{4\pi\varepsilon_0 r}$$

which increases as the separation distance r increases.

Where does this increase in energy go, since we know the total energy must be conserved? It goes into the rest mass energies of the electron and proton in the case of hydrogen. Another way of putting it is that the hydrogen atom has a mass that is **less** than the sum of the separate masses of the electron and proton. The difference in the rest mass energies of the separate objects from the combined system is called the **binding energy**:

$$BE = \left\{ M(\text{separate}) - M(\text{bound}) \right\} c^2$$

In the case of hydrogen, the binding energy is 13.6 eV; that is, hydrogen has a mass that is 13.6 eV less than the sum of the masses of the electron and proton.

Let's consider another example. The deuteron is a bound system of a neutron and a proton. The binding energy is given by:

$$BE = \left\{ M(\text{n}) + M(\text{p}) - M\left({}^2\text{H}\right) \right\} c^2$$

$$BE = \left\{ 939.57 \text{ MeV}/c^2 + 938.28 \text{ MeV}/c^2 - 1875.63 \text{ MeV}/c^2 \right\}$$

$$BE = 2.22 \text{ MeV}$$

Clearly nuclear binding energies are much larger than atomic binding energies! We will explore this more when we study nuclear physics toward the end of this course.

Reaction Energy

Closely related to binding energy is the concept of reaction energy. Not all composite systems have a mass less than the sum of its constituent masses, and some fundamental particles spontaneously decay into particles whose combined mass is less than that of the parent. In these cases, energy is released in the decay or reaction because of the difference in rest mass energies. We define this reaction energy as:

$$Q = \left\{ M(\text{initial products}) - M(\text{final products}) \right\} c^2$$

As you can see, it is just the negative of the binding energy. If Q is positive, we say that the reaction or decay is **exothermic**; that is, it releases energy. If Q is negative, the reaction or decay is **endothermic**; it takes energy to make it happen.

Example: Consider the spontaneous decay of a neutron: $n \to p + e^- + \nu_e$. We can calculate the energy released in this decay by taking the difference in mass of the left-hand side from the right-hand side. The neutrino (ν_e) will be discussed later in the nuclear and particle physics sections; what is relevant here is that its mass is essentially zero.

$$Q = \left\{ M(n) - M(p) - M(e^-) \right\} c^2$$

$$Q = \left\{ 939.57 \text{ MeV}/c^2 - 938.28 \text{ MeV}/c^2 - 0.511 \text{ MeV}/c^2 \right\} c^2$$

$$Q = 0.78 \text{ MeV}$$