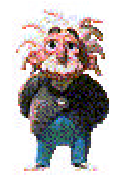
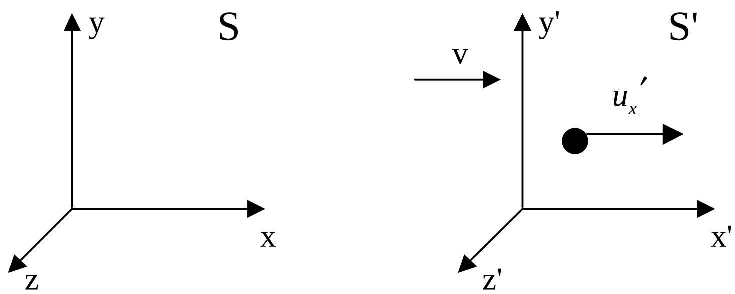


Relativity 3

Disclaimer: These lecture notes are not meant to replace the course textbook. The content may be incomplete. Some topics may be unclear. These notes are only meant to be a study aid and a supplement to your own notes. Please report any inaccuracies to the professor.

Addition of Velocities

Now that we know that the Galilean transformation must be modified, it's time to revisit the topic of adding velocities. Consider two inertial frames S and S' with a relative velocity *v*.



$u_x' = \frac{dx'}{dt'}$ in frame S'

$u_x = u_x' + v$ in a Galilean transformation, which would imply

$u_x > c$ if $u_x' = c$

Consider the inverse Lorentz Transformation:

$x = g(x' + vt')$ and $y = y', z = z'$

$t = g\left(t' + vx'/c^2\right)$

Take differentials:

$dx = g(dx' + vdt')$

$dt = g\left(dt' + v/c^2 dx'\right)$

Divide one by the other:

$$u_x = \frac{dx}{dt} = \frac{dx' + v dt'}{dt' + v/c^2 dx'}$$

$$u_x = \frac{\frac{dx'}{dt'} + v}{1 + v/c^2 \frac{dx'}{dt'}}$$ where we have divided by dt'

Note that $\frac{dx'}{dt'} = u_x{}'$

The velocity addition formulae are:

$$u_x = \frac{u_x{}' + v}{1 + v u_x{}'/c^2}$$

$$u_y = \frac{u_y{}'}{\gamma\left(1 + v u_x{}'/c^2\right)}$$

$$u_z = \frac{u_z{}'}{\gamma\left(1 + v u_x{}'/c^2\right)}$$

Note that even though $y = y'$ and $z = z'$, that $u_y \neq u_y{}'$ and $u_z \neq u_z{}'$

Example: Consider a spacecraft that travels at 0.8*c* from Earth and that launches a projectile with a relative velocity of 0.8*c*. What is the velocity of the projectile from Earth?

Galilean: $u_x = 0.8c + 0.8c = 1.6c > c$!

Lorentz: $u_x = \frac{1.6c}{1 + 0.8^2} = 0.976c < c$

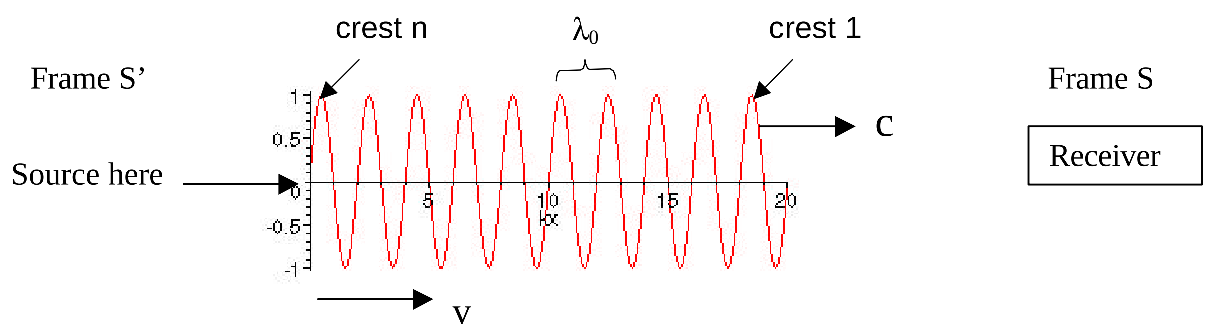
If instead of a projectile we turned on a light beam, both observers on the spacecraft and on Earth would agree that the velocity of the light beam is *c*, as required by Einstein's 2nd postulate.

The addition of velocity formulae tell us that nothing can exceed the speed of light.

Doppler Effect

The Doppler effect is a change in frequency of a traveling wave when the source is moving toward or away from a receiver, like the change in pitch of a car's engine when it travels by you. We can derive the change in "pitch" for light using what we have learned in Special Relativity.

Consider a light wave traveling along the x-axis. It emits *n* wave crests in a time T_0 in the rest frame of the emitter.



Length of wavetrain $= L_0 = n\lambda_0 = cT_0$

$$\Rightarrow \lambda_0 = \frac{cT_0}{n}$$

For light, $\lambda f = c$

$$\Rightarrow f_0 = \frac{c}{\lambda_0} = \frac{n}{T_0} = \text{ frequency of light in rest frame}$$

The frequency is n crests per time T_0

Now consider a receiver in a different inertial frame S. Suppose that the transmitter in frame S' is moving toward the receiver at a velocity *v*. Let's compute the frequency received given that the speed of light is always a constant for all frames.

$L = n\lambda = cT - vT$ length of wavetrain in frame S

$$\Rightarrow \lambda = \frac{c-v}{n}T$$

Distance traveled by crest 1 minus distance source moves by the time of the last crest.

Now from time dilation we know that $T = \gamma T_0$

$$\lambda = \frac{c-v}{n}\gamma T_0$$

$$\Rightarrow f = \frac{cn}{(c-v)\gamma T_0}$$

$$f = \frac{f_0}{\gamma(1-v/c)} \quad \text{Since } f_0 = n/T_0$$

Now we substitute in for γ:

$$f = \frac{\sqrt{1 - v^2/c^2}}{1 - v/c} f_0 = \frac{\sqrt{(1 + v/c)(1 - v/c)}}{1 - v/c} f_0$$

So the Doppler shift equations are:

$$f = f_0 \sqrt{\frac{1 + v/c}{1 - v/c}} \quad \text{Source and receiver approaching}$$

$$f = f_0 \sqrt{\frac{1 - v/c}{1 + v/c}} \quad \text{Source and receiver receding}$$

Thus, when source and receiver approach each other, the frequency is shifted higher. We say that the light is **blue-shifted**.

When source and receiver recede from each other, the frequency is shifted lower. We say that the light is **red-shifted**. Red light has a lower frequency than blue light.

Modern electronics allow us to determine frequencies very accurately, so we can measure relative velocities accurately as well using this effect. Examples include Doppler weather radar, police radar, and even the expansion of the universe!

Lorentz Invariance

We have seen that some quantities change from one inertial frame to another (length, time, velocity, frequency). A quantity which does not change after a Lorentz transformation is said to be **Lorentz Invariant**. One special invariant is the

Space-time Interval:

$$(\Delta s)^2 = (c\Delta t)^2 - \left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right]$$

$$\Delta t = t_2 - t_1$$

$$\Delta x = x_2 - x_1 \quad \text{etc.}$$

This is the generalization of Cartesian distance for 4-dimensional space-time. The same value for Δs is obtained for any inertial frame. So although length and time separately are not invariant from one frame to another, this particular combination is.

We can prove that this is true by applying the Lorentz transformation. For example, consider a subatomic particle which decays in a time τ in its rest frame.

In the rest frame S':

$$t_1' = 0,\ \ t_2' = t\,,\ \ x_1' = x_2' = y_1' = \cdots = 0$$
$$\Rightarrow \Delta s = ct$$

Now make a Lorentz transformation to another frame S moving at velocity *v*:

$$x = g\,(x' + vt') \qquad y' = y = z' = z = 0$$
$$t = g\left(t' + vx'/c^2\right)$$
$$x_2 = gvt\,, \quad x_1 = 0 \quad \Rightarrow \Delta x = gvt$$
$$t_2 = gt\,, \qquad t_1 = 0 \quad \Rightarrow \Delta t = gt$$
$$(\Delta s)^2 = c^2 g^2 t^2 - g^2 v^2 t^2 = g^2 t^2 c^2\left(1 - v^2/c^2\right)$$
$$= \frac{c^2 t^2}{1 - v^2/c^2}\left(1 - v^2/c^2\right)$$
$$\Rightarrow \Delta s = ct \text{ as in the rest frame}$$

Some terminology:

$\Delta s^2 > 0 \quad \Rightarrow \quad$ time-like

A frame exists where 2 events occur in one place, separated by time.

$\Delta s^2 = 0 \quad \Rightarrow \quad$ light-like

2 events are separated by the speed of light.

$\Delta s^2 < 0 \quad \Rightarrow \quad$ space-like

No light signal can connect the 2 events.