General Relativity

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The Equivalence Principle

Einstein’s Special Theory of Relativity, published in 1905, described the transformation between inertial frames with a constant relative velocity. What about the transformation between frames with a relative acceleration? A frame undergoing acceleration feels a force (from Newton’s 2nd Law), so one can definitely distinguish it from a frame moving at a constant velocity. In addition, can we incorporate a theory of gravity into Relativity; that is, modify Newton’s theory of gravity so that it obeys relativistic transformations? Einstein accomplished both goals in his General Theory of Relativity, published in 1916. The heart of this theory is the Principle of Equivalence. Although we won’t discuss the mathematics of General Relativity, the Equivalence Principle is a very powerful tool that allows us to make some calculations.

Newton’s 2nd Law reads: \( F = m \cdot a \)

This law tells us that the acceleration of an object is proportional to the force acting on a body divided by the amount of inertia (resistance to motion) it contains. Inertia is just mass, which we label \( m_i \). Now let’s consider some fundamental forces which can accelerate an object.

Coulomb’s Law is: \[ F = \frac{Q_1 Q_2}{4\pi\varepsilon_o r^2} \] where \( \frac{1}{4\pi\varepsilon_o} = 8.98755 \times 10^9 \) N - m\(^2\) / C\(^2\)

The force is proportional to the product of the electric charges of two bodies.

Newton’s Law of Gravity is:
\[ F = G \frac{m_1 m_2}{r^2} \] where \( G = 6.6726 \times 10^{-11} \) N - m\(^2\) / kg\(^2\)

Apparently, the “charge” of gravitation is mass. On the surface of the Earth, we can write this equation as:
\[ F = m \cdot g \text{ where } g = 9.8 \text{ m/s}^2 \]

Notice that this equation would be exactly the same as Newton’s 2nd Law if the gravitational “charge” equals inertial mass \( m_i = m_g \) and if the acceleration of a body is the same as the acceleration due to gravity (\( a = g \)). The experienced force would be
identical. Is it identical? As far as we know, it is identical to a very high degree of accuracy from many precise measurements made by laboratory experiments, satellite measurements, and astronomical measurements. This forms the basis of another of Einstein’s postulates:

**Equivalence Principle:**

*There is no experiment done in a small confined space which can tell the difference between a uniform gravitational field and an equivalent acceleration.*

Why the “charge” of gravity is the inertia of a body is something of a mystery. It didn’t have to be that way. There could have been some other charge like electricity that caused gravitation. Instead, there is some deep intimate connection. What Einstein is basically saying is this: If you cannot tell the difference between two forces, they must be the same force. So somehow gravity curves space to make objects fall (accelerate).

The Equivalence Principle allows us to make calculations involving gravity by considering instead reference frames undergoing acceleration. By applying the Equivalence Principle, the answer equally applies to a reference frame with the same gravitational acceleration. Let’s see how:

**Bending of Light Example:**

Consider a rocket undergoing constant acceleration. Light entering one side of the rocket will appear to “fall” toward the floor as it crosses the rocket. This is because it takes a finite amount of time for light to cross the rocket, and in that time, the rocket has moved. Since it is undergoing constant acceleration, the trajectory of the light will be a parabola.

Now by the Equivalence Principle, this result applies as well to a stationary frame with a gravitational acceleration equal to the previous rocket acceleration. So light “falls” in a gravitational field. This is quite remarkable given that the particles of light “photons” are massless! Newton’s Law would have predicted no bending.

How much does light bend? Let the width of the rocket be $L$:

$$ t = \frac{L}{c} \quad \text{Time to cross rocket ship} $$

$$ s = \frac{1}{2} at^2 = \frac{1}{2} a \frac{L^2}{c^2} \quad \text{Amount deflected vertically} $$

Now for $a = g = 9.8 \, \text{m/s}^2$ as on the surface of the Earth and $L = 3000 \, \text{km}$, the length of a continent on Earth

$$ s = \frac{1}{2} \, \text{mm} $$
So the deflection is very small. However, when light enters a stronger gravitational field, such as that of the Sun, the deflection is more. Einstein made a spectacular prediction that light passing by the Sun during a 1919 eclipse would cause the apparent position of the star to move by 1.75 seconds of arc. Two teams made a measurement whose average value was $1.9 \pm 0.2$ arc-seconds, confirming the prediction of General Relativity. This success catapulted Einstein into celebrity status!

**Gravitational Doppler Effect**

Another striking prediction based on the Equivalence Principle is that the frequency of light changes when entering or leaving a gravitational field. Again, we turn to solving the problem in an accelerating reference frame, then apply it to a stationary frame with a gravitational field.

Consider light traveling from the bottom to the top of a rocket undergoing constant acceleration. Let point A be at the bottom of the spacecraft, and point B at the top. The separation distance measured in the reference frame of the rocket is $H$. When light first leaves point A, the velocity of the rocket is $v_A$ with respect to another reference frame (the Earth, for example). It takes a time $T$ for light to travel to point B, so:

\[ v_A = \text{velocity of rocket when light is emitted} \]
\[ v_B = v_A + aT = \text{velocity of rocket when light received} \]

The time $T$ for light to reach B is:

\[ T = \frac{H + \delta}{c} \approx \frac{H}{c} \quad \text{we assume that the acceleration is "small"} \]

The change in velocity of the rocket between emission and reception is:

\[ \Delta v = v_B - v_A = aT = \frac{aH}{c} \]

**Point B is receding from point A at a relative velocity of** $v = \frac{aH}{c}$

This is a strange way of thinking about it, because in the reference frame of the rocket there is a fixed distance between the floor and the ceiling. However, from the perspective of an outside inertial frame (the Earth) which is not undergoing acceleration, what is observed is light leaving point A (at a velocity of $c$) and reaching point B sometime later. The receiver is moving away from the light at an additional relative velocity of $v$. In this external frame, we can use the results of Einstein’s Special Theory of Relativity. Specifically, for a relative receding velocity, the frequency of the light is Doppler-shifted:

\[ f = \frac{1-v/c}{1+v/c} f_0 = \frac{(1-v/c)(1-v/c)}{(1+v/c)(1-v/c)} f_0 = \gamma(1-v/c)f_0 \]

Doppler shift for receding velocities
For relative velocities small compared to the velocity of light (corresponding to small acceleration of the rocket), we can make the approximation:

\[
f = \left(1 - \frac{v}{c}\right) f_0
\]

\[
\Rightarrow \frac{\Delta f}{f} \approx -\frac{\Delta v}{c} = -\frac{aH}{c^2}
\]

But the Equivalence Principle tells us that the same result applies for the situation when \( a = g \), the acceleration due to gravity:

\[
\frac{\Delta f}{f} \approx -\frac{gH}{c^2}
\]

The minus sign means that the frequency decreases (red-shifted). This result applies to the case when the gravitational field does not vary over the height \( H \), such as close the Earth’s surface.

An alternate way of deriving this result is to apply conservation of energy, and make use of the fact that light is composed of photons, each with an energy of

\[
E = hf
\]

Since \( E_0 = mc^2 \) is the rest energy of an object according to the Special Theory of Relativity, the photon “mass” is \( m = E_0 / c^2 = hf_0 / c^2 \). The constant \( h \) is Planck’s Constant.

Photons will be discussed a little later in the course. In any case, if a photon travels vertically, it must lose kinetic and rest energy as it gains gravitational potential energy:

\[
E = hf = E_0 - mgH = E_0 \left(1 - \frac{gH}{c^2}\right) = hf_0 \left(1 - \frac{gH}{c^2}\right)
\]

So the frequency shift is the same as derived above.

**Experimental Verification:** This Doppler shift was tested by Pound&Rebka in 1960 using \( \gamma \)-rays. From a tower of height 20 m, they measured a shift of

\[
\frac{\Delta f}{f} = 2 \times 10^{-15}
\]

This is a very small and very difficult effect to measure, but it was nevertheless verified by making use of some interesting quantum-mechanical properties of detectors.
Frequency is just the inverse of period, so we can write:

\[ f = \frac{1}{T} \]

\[ \frac{\Delta f}{f} = \frac{f - f_0}{f_0} = \frac{1}{T} - \frac{1}{T_0} = \frac{T_0 - T}{1/T_0} = \frac{\Delta T}{T} \]

So the Doppler Effect also implies time dilation:

\[ \frac{T_0 - T}{T} = 1 - \frac{gH}{c^2} \]

\[ T_0 \approx \frac{T_0}{1 - \frac{gH}{c^2}} \approx \left( 1 + \frac{gH}{c^2} \right) T_0 \]

In other words, the clocks at the bottom of a gravitational well run slower as viewed from clocks at the top of a well.

**Example:** Your head ages faster than your feet! We can compute the time difference over a lifetime:

\[ H = 2 \text{ m} \]

\[ T = 70 \text{ years} = 2.2 \times 10^9 \text{ s} \]

\[ \Delta T = \frac{gH}{c^2} T = \frac{(2 \text{ m})(9.8 \text{ m/s}^2)}{(3 \times 10^8 \text{ m/s})^2} - 2.2 \times 10^9 \text{ s} \]

\[ \Delta T = 5 \times 10^{-7} \text{ s} = 500 \text{ ns} \]

Unfortunately, not exactly a Fountain of Youth!

We derived the Doppler Effect for a uniform gravitational field. However, it is useful to know the formula for the case when the gravitational field changes significantly between two points (i.e. from the surface of the Sun to the next star). In that case, the correct formula is:

\[ \frac{\Delta f}{f} = -\frac{GM}{c^2} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \]

**General Gravitational Red-shift**

The radius from the center of the gravitational field is denoted by \( r \). \( G \) is Newton’s gravitational constant, and \( M \) is the mass of the gravitating object. The acceleration due to gravity is related to these constants by: \( g = \frac{GM}{r^2} \)
We can now determine the Gravitational Red-shift for light emitted by several astronomical sources and detected on Earth. In that case \( r_1 \) is the radius of the source and \( r_2 \) is infinite to a good approximation.

**White Dwarf**

This is a star that has burned all its nuclear fuel and has contracted under its own gravity.

\[
R = 5000 \text{ km} = 5 \times 10^6 \text{ m} \quad \text{(about the radius of the Earth)}
\]
\[
\rho = 10^9 \text{ kg/m}^3 \quad \text{(a density a million times that of water)}
\]
\[
M = \frac{4}{3} \pi R^3 = 5 \times 10^{29} \text{ kg} \quad \text{(about 1/4 the mass of our Sun)}
\]
\[
\frac{\Delta f}{f} = -\frac{GM}{c^2 R} = -\frac{(6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2})(5 \times 10^{29} \text{ kg})}{(3 \times 10^8 \text{ m/s})^2 (5 \times 10^6 \text{ m})}
\]
\[
\frac{\Delta f}{f} = -8 \times 10^{-5} = 0.01\%
\]

Thus, the shift is 10 billion times larger than from the surface of the Earth. By the way, the gravitational acceleration on the surface of a white dwarf is \( 1.4 \times 10^8 \text{ m/s}^2 \)!

**Neutron Star**

This is a star that has collapsed after a supernova explosion. The gravitational pressure is so great that atoms collapse as well, and the electron are swallowed up by protons, converting everything into neutrons. A neutron star is a colossal nucleus!

\[
R = 10 \text{ km}
\]
\[
\rho = 5 \times 10^{17} \text{ kg/m}^3 \quad \text{(nuclear density)}
\]
\[
M = \frac{4}{3} \pi R^3 = 2 \times 10^{30} \text{ kg} \quad \text{(about the mass of our Sun)}
\]
\[
\frac{\Delta f}{f} = -\frac{GM}{c^2 R} = -\frac{(6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2})(2 \times 10^{30} \text{ kg})}{(3 \times 10^8 \text{ m/s})^2 (10^4 \text{ m})}
\]
\[
\frac{\Delta f}{f} = -15\%
\]

Thus, clocks run 15% slower on the surface of a neutron star. The gravitational acceleration on the surface of a neutron star is \( 10^{12} \text{ m/s}^2 \)!
Black Hole

This is a star that has collapsed even farther than a neutron star. The gravitational pressure is so great that not only do atoms collapse, but the nucleus does as well! A collapsing star becomes a black hole if its radius satisfies:

\[ R < \frac{2GM}{c^2} \]

= Schwarzschild radius

So the gravitational red-shift is:

\[ \frac{\Delta f}{f} = -\frac{GM}{c^2 R} = -\frac{1}{2} \]

However, this answer is only approximate, because we derived the Doppler effect assuming a weak gravitational field, and that is not the case for a black hole! For those students who want to know the correct result from the full theory of General Relativity:

\[ \frac{\Delta f}{f} = 1 - \frac{1}{\sqrt{1 - \frac{2GM}{c^2 R}}} \]

So as the decreases to the Schwarzschild radius, the frequency shift goes to negative infinity. The lower limit can only be –1, so what we are observing is that

**Light cannot escape from the black hole**

Light loses all its energy before escaping, and so “falls” back inward. A black hole is black because light does not escape. Since nothing can travel faster than the speed of light, the inside of a black hole is effectively cut off from the known universe.

Gravitational Waves

The Newtonian Theory of gravity essentially tells us that gravity acts instantaneously between two masses, even if the two objects are stars separated by light-years. Relativity tells us that nothing can travel faster than the speed of light, so there is a problem here. However, just as Maxwell’s equations predict electromagnetic waves, Einstein’s equations of General Relativity predict gravitational waves that travel at the speed of light. Thus, two stars cannot interact instantaneously. The waves must first reach the other object. This is a firm prediction of General Relativity.

Gravity, though, is the weakest off all forces. We have seen how the effects of General Relativity on the surface of the Earth are incredibly small. Because of this, no experiment has yet detected gravitational waves.
Nevertheless, indirect evidence for gravitational waves exists. Hulse and Taylor won the Nobel prize for their measurements on the orbital frequency of a binary neutron star system. Some neutron stars are “pulsars”, and they emit radiation at rapid intervals corresponding to the rapid rotation rate. Such a binary system is predicted to emit gravitational waves, and such waves carry away energy from the system. The orbital distance must then contract to conserve energy, and this increases the orbital frequency. The measurements of this increase for pulsar PSR 1913+16 is in good agreement with the prediction of General Relativity.

However, we may soon be on the verge of direct detection of gravitational waves. An experiment called LIGO (Laser Interferometer Gravitational Observatory) is under construction in the U.S., as are a couple of experiments in Europe. LIGO is a pair of interferometers (one in Washington and one in Louisiana). The length of one arm of the interferometer is 4 km! Despite this, the experimental groups must look for changes in the arm length distance as a gravitational waves passes on the order of $10^{-18}$ m, which is 1000 times smaller than the diameter of a proton! The University of Florida is one of the major partners in this exciting endeavor.