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PHZ4390
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Homework 5

Due Monday, Sep. 30

1. A particle of mass m is in a one dimensional "box" of length L with infinitely high potential walls, i.e., $V(x)=0$ when $0 \le x \le L$ and $V(x)=\infty$ otherwise. Classically, a particle in such a box with some energy E (all kinetic) would bounce back and forth between the walls.
   a. (1 pt) Write down Schrodinger's equation for an eigenstate of energy E in the region $0 \le x \le L$.
   b. (1 pt) Show that $\psi(x)=A\cos(kx)+B\sin(kx)$ is a general solution to the equation (suppressing the time dependence for now). What is k in terms of E?
   c. (2 pts) Show that the boundary conditions imply $A = 0$ and $k = n\pi / L$, where n is a positive integer.
   d. (2 pts) What are the allowed energies?
   e. (2 pts) Show that the normalized spatial solutions (total probability = 1) can be written $\psi_n(x)=\sqrt{2/L}\sin(n\pi x/L)$, where n is a positive integer. Why don't we include negative integers in the list of possible solutions?
   f. (2 pts) Show that any two solutions $\psi_n(x)$ and $\psi_m(x)$ are orthonormal, i.e. $\int_{-\infty}^{\infty}\psi_n^*(x)\psi_m(x)dx=\delta_{nm}$.
   g. (2 pts) What is the form of a general energy eigenstate, including time dependence?
2. (Bonus: 3 pts) Plot using wxMaxima or the tool of your choice the probability distribution $|\psi(x)|^2$ of the ground state and the 2 next excited states of the particle in the 1-D box.
3. (5 pts) What is the probability of a particle in the ground state being located somewhere in the interval $\frac{1}{2}L \le x \le \frac{4}{5}L$?

4. Consider a particle in a 3-D cube of sides L with one corner at the origin and sides parallel to the x, y, z axes. The potential energy is 0 in the cube and positive infinite outside it. This is called the 3-D box, a generalization of the 1-D box you considered in problem 1.

    a. (2 pts) Write down Schrodinger's equation for an eigenstate of energy E in the region $0 \le (x,y,z) \le L$. This is just a generalization of the 1-D equation.

    b. (2 pts) Show briefly that the normalized 3-D spatial solutions (total probability = 1) are $\psi(x) = \sqrt{8/V}\sin(n_x \pi x/L)\sin(n_y \pi y/L)\sin(n_z \pi z/L)$, using 3 integers n_x, n_y, and n_z, and the volume V. This is easy if you worked out problem 1.

    c. (2 pts) What is the allowed energy for each solution?

    d. (2 pts) Which solutions are "degenerate" (identical energies for different solutions)?

    e. (2 pts) What are the 6 lowest energies and their degeneracies?

5. According to the Heisenberg uncertainty principle, $\Delta x \, \Delta p_x \ge \frac{1}{2}\hbar$ where Δx and Δp_x are the uncertainties in the measurements of x and p_x, respectively. Physicists typically approximate one of these uncertainties to get an estimate of the other. A typical problem involves estimating the ground state energy for a particle moving within a small spatial volume.

    a. (4 pts) Make a good "guesstimate" for the uncertainty of the particle's position in the 1-D box to estimate its momentum uncertainty and use this to calculate an approximate ground state energy assuming non-relativistic motion. How does this compare to the ground state energy determined in problem 1(d)?

    b. (3 pts) Use the uncertainty principle in 3-D (i.e., use all three dimensions) to estimate the kinetic energy (in MeV) of a proton that is confined to a nucleus of radius 2 fm. How does this compare to the typical binding energy of a nucleon?

    c. (Bonus 3 pts) Use the uncertainty principle in 3-D to estimate the kinetic energy (in MeV) of an electron that is confined to a nucleon of radius 1 fm. (Note that it is ultra-relativistic so the mass can be ignored.) This problem has historical significance.