# TIME DEPENDENCE IN TWO DIMENSIONAL STRING THEORY 

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- It has been traditionally difficult to describe time dependent backgrounds in string theory - partly because of the absence of an off-shell formulation.
- We have to understand the physics of such backgrounds well enough to be able to address questions in cosmology as well as in the evolution of black holes.
- One would expect that a holographic formulation would be useful. However for the most interesting concrete realizations of holography we do not have explicit dual transformations.
- In this talk, I will discuss some of the issues in a toy model - the two dimensional noncritical string - where we have an explicit holographic map.
- One of our main aims is to understand the emergence of space-time.


## 1 THE $c=1$ MATRIX MODEL

We will deal with 2d string theories whose holographic description is double scaled Gauged Matrix Quantum Mechanics - a gauge theory in $0+1$ dimensions.

$$
H=\frac{1}{2} \sum_{i j}\left[-\frac{\partial^{2}}{\partial M_{i j}^{2}}-M_{i j} M_{j i}\right]
$$

where $M$ is a $N \times N$ hermitian matrix. $N \rightarrow \infty$

In addition there is a constraint which restricts to singlet states

In the singlet sector this is a theory of $N$ fermions in 1 space dimension with the single particle hamiltonian

$$
h=\frac{1}{2}\left(p^{2}-x^{2}\right)
$$

and the ground state is the filled fermi sea

The scaled fermi energy is $-\mu$ which is held fixed in the double scaling limit


Figure 1: The fermi surface in phase space - ground state

- Modern interpretation : $M_{i j}$ are open string degrees of freedom of $N$ D0 branes
(McGreevy and Verlinde (2003);
Klebanov, Maldacena and Seiberg (2003))
- Small fluctuations of the fermi surface are massless bosons - - these are the closed string degrees of freedom.
- These are linear combinations of NSNS and RR bosons of 2d Type 0B string theory
(Douglas, Klebanov, Kutasov, Maldacena, Martinec, Seiberg (2003);

Takayanagi and Toumbas (2003))

## 2 COLLECTIVE FIELD THEORY AND STRINGS

The massless bosons are in fact the fluctuations of the density of eigenvalues of the matrix $M$

$$
\rho(x, t)=\partial_{x} \phi=\frac{1}{N} \operatorname{Tr} \delta(x \cdot I-M(t))
$$

This is the basic holographic map.
(S.R.Das and A. Jevicki, 1990)

The classical action for $\phi$ is given by

$$
S=\int d t d x\left[\frac{\left(\partial_{t} \phi\right)^{2}}{2 \partial_{x} \phi}-\frac{\pi^{2}}{6}\left(\partial_{x} \phi\right)^{3}+\left(\frac{1}{2} x^{2}-\mu\right) \partial_{x} \phi\right]
$$

- The ground state corresponds to the lowest energy classical solution

$$
\partial_{x} \phi_{0}=\frac{1}{\pi} \sqrt{x^{2}-2 \mu}
$$

- Fluctuations are represented by two bosonic fields $\eta_{L, R}$ which do not talk to each other at the perturbative level. They are massless fields moving in a metric which is conformal
to

$$
d s^{2}=-d t^{2}+\frac{d x^{2}}{x^{2}-2 \mu}
$$

- The edges of the eigenvalue distribution $x=$ $\pm \sqrt{2 \mu}$ act as reflecting mirrors for $\eta_{R, L}$ respectively.
- The fluctuations are weakly coupled in the large $|x|$ region. They are strongly coupled near the mirror.
- It is useful to introduce Minkowskian coordinates $y$

$$
\begin{aligned}
x & = \pm \sqrt{2 \mu} \cosh y \\
d s^{2} & =-d t^{2}+d y^{2}
\end{aligned}
$$

- Space of the string theory arises from the space of eigenvalues.

However the "space" which appears in the worldsheet action is related nonlocally to this space $(x$ or $Q)$.

## 3 MATRIX COSMOLOGIES

Since we have an off shell formulation of the holographic theory as well as its "gravity" dual we can consider time dependent backgrounds which are large deformations of the ground state

Such backgrounds have been known for a while : (Minic, Polchinski and Yang;
Moore and Plesser;
Alexandrov, Kazakov and Kostov))
Recently they have been proposed as models of cosmology
(Karczmarek and Strominger)

We want to investigate

1. Emergence of space-time in these backgrounds
2. Particle production
3. Conversion of "left" type particles into "right" type particles

## References:

1. J. Karczmarek and A. Strominger, hep-th/0309138
2. J. Karczmarek and A. Strominger, hep-th/0403169
3. S.R. Das, J. Davis, F. Larsen and $P$. Mukhopadhyay, hep-th/0403275
4. P. Mukhopadhyay, hep-th/0406029
5. S.R. Das and J. Karczmarek-in progress

## 4 GENERATING SOLUTIONS

The matrix model action has an infinite number of global symmetries - $W_{\infty}$. In the single particle phase space the generators are

$$
w_{r s}=\frac{1}{2} e^{(r-s) t}(x-p)^{r}(x+p)^{s}
$$

For $r \neq s$ these charges do not commute with the hamiltonian $w_{11}$

Starting from the ground state they then generate nontrivial time dependent solutions

Easy to write down finite transformations for charges of the form $w_{0 r}$ or $w_{s 0}$

$$
\begin{array}{ll}
w_{r 0}: & x^{\prime}=x+\lambda r e^{r t}(x-p)^{r-1} \\
& p^{\prime}=p+\lambda r e^{r t}(x-p)^{r-1}
\end{array}
$$

These take the ground state Fermi surface into

$$
\frac{1}{2}\left(x^{2}-p^{2}\right)+\lambda r e^{+r t}(x-p)^{r}=\mu
$$

These solutions reduce to the ground state at early times.

## 5 DRAINING AND FLOODING FERMI SEAS

For $r=1$ the phase space transformations are in fact coordinate transformations

Fluctuations which start on the left side perceive a mirror which is approaching towards the asymptotic region and the whole universe effectively shrinks

Fluctuations which start on the right side perceive a mirror which is receding away


Figure 2:

Not all classical solutions of the fermionic theory correspond to classical solutions of collective field theory

However Fermi surface profiles which are quadratic do correspond to classical collective field solutions

Non-quadratic profiles generically signify states whose quantum dispersions are large
(S.R. Das, hep-th/0401067)
$W_{10}$ solutions have quadratic profiles. They correspond to time dependent classical solutions $\phi_{0}(x, t)$

$$
\begin{aligned}
\partial_{x} \phi_{0} & =\frac{1}{\pi} \sqrt{\left(x+\lambda e^{t}\right)^{2}-2 \mu} \\
\partial_{t} \phi_{0} & =\partial_{x} \phi_{0}(x, t) \lambda e^{t}
\end{aligned}
$$

Fluctuations around any classical solution are massless scalars living in a metric which is conformal to

$$
d s^{2}=-d t^{2}+\frac{\left(d x+\frac{\partial_{t} \phi_{0}}{\partial_{x} \phi_{0}} d t\right)^{2}}{\left(\pi \partial_{x} \phi_{0}\right)^{2}}
$$

and once again we have two such scalars for the two sides of the Fermi sea.

As for any metric in two dimensions we can go to Minkowskian coordinates $(\tau, \sigma)$ and solve for the linearized modes

We will henceforth set $2 \mu=1$

$$
\begin{array}{rl}
R: \tau=t & x=\cosh \sigma-\lambda e^{\tau} \\
L: \tau=t & x=-\cosh \sigma-\lambda e^{\tau}
\end{array}
$$

In these coordinates the mirror is always at $\sigma=0$.

$$
d s^{2}=-d \tau^{2}+d \sigma^{2}
$$

Null lines are $\tau \pm \sigma=0$.
In a Penrose diagram fluctuations start from

$$
\mathcal{I}^{-}: \sigma_{-}=\sigma-\tau=\infty
$$

get reflected by the mirror at $\sigma=0$ and emerge at

$$
\mathcal{I}^{+}: \sigma_{+}=\sigma+\tau=\infty
$$



Figure 3:

What happens in the $x$ space is, however quite different.

To find the trajectory of an incoming or reflected pulse substitute $\sigma= \pm\left(\tau-\tau_{0}\right)$ in equation defining $\sigma$

$$
\begin{aligned}
x(t) & =\cosh \sigma-\lambda e^{\tau} \\
& \rightarrow \cosh \left(\tau-\tau_{0}\right)-\lambda e^{\tau}
\end{aligned}
$$

This should give the trajectory of a localized pulse in $x$ space at the linearized level

Interestingly, the resulting $x(t)$ gives in fact the EXACT classical trajectory of a point of the fermi surface as calculated from the fermion theory.

Define for L fluctuations

$$
x=-\cosh Q
$$

L fluctuations therefore stay on the L side.


Figure 4:

In 0B interpretation, L fluctuations correspond to equal amounts of NSNS and RR scalars - time evolution does not change this balance.

At late times the null rays can be characterized by

$$
Q-t=\mathrm{constant}
$$

Define for R fluctuations $x=\sinh Q$. The value


Figure 5:
of $\tau_{0}$ determines whether these fluctuations will actually get "reflected" in $x$ space.

For $\tau_{0}>0$ they spill over to the other side

In the $0 B$ interpretation this means that as time evolves these fluctuations get convereted into opposite amounts of NSNS and RR scalars.

At late times null rays are characterized by

$$
Q \pm t=\text { constant }
$$

## 6 PARTICLE PRODUCTION AND STRINGY EFFECTS

Consider first the L fluctuations. Our problem is then pretty much like that of moving mirror - except the mirror is now coming towards the asymptotic region

Nevertheless similar physics will lead to particle production.

The "in" vacuum is the vacuum defined in terms of modes which are plane waves in $(\sigma, \tau)$ coordinates

At late times we may characterize $\mathcal{I}^{+}$as usual in terms of $(\sigma, \tau)$. However on $\mathcal{I}^{+}$

$$
\sigma^{-}=Q^{-}+\log \left[1-\tilde{\lambda} e^{-Q^{-}}\right]
$$

so that $Q^{-}$parametrizes $\mathcal{I}^{+}$equally well

If we define

$$
T_{Q^{-} Q^{-}}=\left.\left(\partial_{Q^{-}} \eta\right)\right|_{\sigma^{+}}
$$

the energy flux of produced particles is calculable using the anomaly relation

$$
\begin{aligned}
<T_{Q^{-}, Q^{-}}>_{\text {in }}= & \left(\frac{\partial \sigma^{-}}{\partial Q^{-}}\right)^{2}<T_{\sigma^{-}, \sigma^{-}}>_{\text {in }} \\
& +\frac{1}{24 \pi}\left\{\sigma^{-}, Q^{-}\right\}_{S}
\end{aligned}
$$

Normally we would put $<T_{\sigma^{-}, \sigma^{-}}>_{\text {in }}=0$ and the entire answer would be given by the Schwarzian term. This evaluates to

$$
\frac{1}{24 \pi} \frac{\lambda e^{-Q^{-}}\left(1-\frac{1}{2} \lambda e^{-Q^{-}}\right)}{\left(1-\lambda e^{-Q^{-}}\right)^{2}}
$$

This diverges at the point where the mirror hits $Q^{+}=\infty$


Figure 6:

## In our case this is not correct

This has got to do with the fact that we are actually dealing with a string theory

In a string theory, there are no UV divergences in quantities like the energy and therefore no normal ordering ambiguity

For example, the one loop correction to the ground state energy is given by the torus diagram which is finite

This should mean that in our calculation we cannot arbitrarily normal order and arrange for

$$
<T_{\sigma^{-}, \sigma^{-}}>_{\text {in }}=0
$$

Rather we should have a finite result without any need for normal ordering

How could this happen in a field theory ?

Collective field theory is not really a conventional field theory.

A careful derivation from the matrix model reveals that there are singular terms of order $\hbar$ in the action

For our case the relevant term is

$$
\Delta S=\frac{1}{2} \int d t d x \partial_{x} \phi\left[\partial_{x} \partial_{x^{\prime}} \log \left|x-x^{\prime}\right|\right]_{x=x^{\prime}}
$$

Thus at one loop quantities like the ground state energy has two contributions

1. The contribution from integrating out fluctuations. This is the usual quadratically divergent term.
2. The contribution from this explicit singular term in the action.

In the sum, the singular conmtribution cancels exactly and one is left with a finite answer.

$$
<T_{\sigma^{-}, \sigma^{-}}>_{i n}=<T_{\sigma^{+}, \sigma^{+}}>_{i n}=-\frac{1}{48 \pi}
$$

which is the same answer obtained from fermionic theory or from the torus diagram on the worldsheet.

For our time dependent background, we have to use this ground state em tensor in the anomaly equation

$$
\begin{aligned}
<T_{Q^{-}, Q^{-}}>_{\text {in }}= & \left(\frac{\partial \sigma^{-}}{\partial Q^{-}}\right)^{2}<T_{\sigma^{-}, \sigma^{-}}>_{\text {in }} \\
& +\frac{1}{24 \pi}\left\{\sigma^{-}, Q^{-}\right\}_{S}
\end{aligned}
$$

The result is remarkable : there is a partial cancellation between this and the Schwarzian term leading to

$$
<T_{Q^{-}, Q^{-}}>_{\text {in }}=-\frac{1}{48 \pi}
$$

exactly as in the ground state!
(Das, Davis, Larsen and Mukhopadhyay)

Working on the R branch, the story is similar except that in the 0B interpretation, $\mathcal{I}^{+}$has two parts - on one side one has L type particles and on the other side R type particles.

Note that we have used coordinates $\left(\sigma^{+}, Q^{-}\right)$ and not $\left(Q^{ \pm}\right)$since the generator of $\mathcal{I}^{+}$is $\left.\frac{\partial}{\partial Q^{-}}\right|_{\sigma^{+}}$

If one uses canonical definitions of the em tensor particle production can be calculated directly from the expressions and carefully regularizing the answer. There is a cancellation, but not complete.
(Mukhopadhyay)
The singular term in the collective field theory is a signature of the fact that we are dealing with a string theory. This gets reflected in the final answer for particle production.

## The opening hyperbola solution

We now discuss some aspects of another solution in which the mirror seems to disappear after a finite time and the emergence of space from the matrix is much more complicated.
(S.R. Das and J. Karczmarek - to appear)

The fermi surface is now given by

$$
\frac{1}{2}\left(x^{2}-p^{2}\right)+\lambda e^{2 t}(x-p)^{2}=\mu
$$



Figure 7:

This is again a quadratic profile and corresponds to a classical solution in collective field theory

$$
\partial_{x} \phi_{0}=\frac{1}{1-e^{2 t}} \sqrt{x^{2}-\left(1-e^{2 t}\right)}
$$

which diverges at $t=0$
In fact the classical energy density of this solution diverges everywhere at the time $t=0$

It is clear that small ripples on the fermi sea propagate perfectly smoothly across $t=0$. With $x=\sinh Q$


Figure 8: Null lines in $t-Q$ space

Fluctuations are massless particles

Miknowskian coordinates $(\tau, \sigma)$
For $t<0$

$$
\begin{aligned}
x & =\frac{\cosh \sigma}{\sqrt{1+e^{2 \tau}}} \\
e^{t} & =\frac{e^{\tau}}{\sqrt{1+e^{2 \tau}}}
\end{aligned}
$$

There is a mirror, which is always at $\sigma=0$

$$
-\infty<t<0 \leftrightarrow-\infty<\tau<\infty
$$

However $\tau=\infty$ is not the end of time

For $t>0$

$$
\begin{aligned}
x & =\frac{\sinh \sigma}{\sqrt{e^{-2 \tau}-1}} \\
e^{t} & =\frac{e^{-\tau}}{\sqrt{e^{-2 \tau}-1}}
\end{aligned}
$$

and there is no mirror anymore.

$$
0<t<\infty \leftrightarrow-\infty<\tau<0
$$

These $(\tau, \sigma)$ coordinates are not yet the physical space and time of the string theory defined through the worldsheet

However they are useful to determine the properties of the space-time generated


Figure 9:
In the $(\tau, \sigma)$ space all incoming rays get reflected by the mirror at $\sigma=0$ ending at what one would normally think of as " $\mathcal{I}^{+}$", viz. $\sigma+\tau=\infty, \sigma_{-}=$ finite

However matrix model tells us to attach another copy of Minkowskii space along this " $\mathcal{I}^{+}$"

This second copy is half of a Minkowski space and ends along the spacelike line $\tau=0$

The matrix model does not seem to allow going beyond $\tau=0$ though that is what one would normally do


Figure 10: Null lines in $Q-t$ space
This is because in terms of the time of the matrix model, fluctuations have already travelled for an inifnite amount of time

# In fact the space of eigenvalues do not have spacelike properties for $t>0$ 



Figure 11: Constant $x$ curves in $\tau-\sigma$ space for $t<0$. The blue line has $x=1$. The green line has $x=2$


Figure 12: Constant $x$ curves in $\tau-\sigma$ space for $t>0$. The blue line has $x= \pm 1$. The green line has $x= \pm 2$


Figure 13: Constant $Q+t$ curves in $\tau-\sigma$ space for $t>0$. The green, blue and red lines have $e^{Q+t}=10,100,1000$ respectively
In a similar way large values of $Q+t$ become the $\tau=0$ axis - a spacelike line

One can go ahead and calculate energy momentum tensors, though one needs to perform an explciit calculation rather than use an anomaly argument.

## 7 OUTLOOK

So far as fluctuations around the ground state are concerned, the eigenvalues become a spacelike direction.

However in a strongly time-dependent background this is more subtle

This has consequences for the relationship to the spacetime which appears as the target space of worldsheet via a "leg pole" transform.

It is not clear how to determine the correct leg pole factor in this case

The only way we know how to do this is to compare with worldsheet calculations.

As a first step, we have been able to come up with a proposal for the worldsheet perturbation for these solutions.

The main lesson from these studies is that the holographic correspondence may "manufacture" space-time in a rather nontrivial way.

In some examples, a straightforward time evolution in a time dependent hamiltonian becomes re-interpreted as particle production in the string theory

In other cases, matrix eigenvalues cease to behave as spacelike coordinates and the correspondence is even less clear.

