High Energy Scattering in 2D String Theory

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based on hep-th/0407136 (w/ Kazumi Okuyama)

- The 2D string and its matrix model
- The exact S-matrix
- The number distribution in high-energy scattering
- Whither the black hole?

Introduction

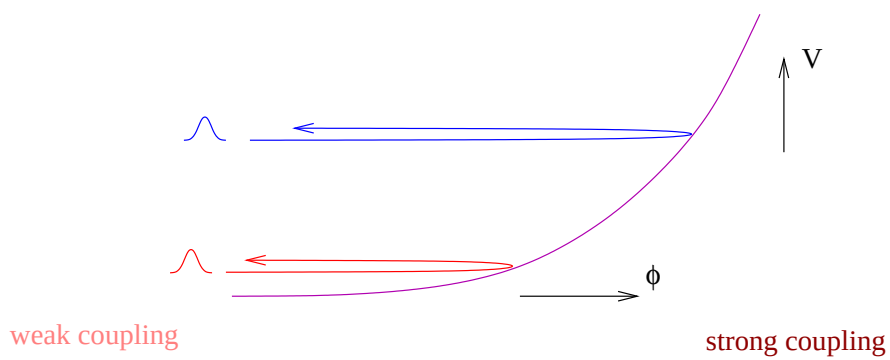
2D string theory illuminates two topics of recent investigations, providing

- A solvable example of open/closed string duality
- A laboratory for the study of tachyon condensation

The 2D closed string background

$$ds^2 = d\phi^2 - dx^2 \quad , \qquad \Phi = \tfrac{1}{2} Q\phi \quad , \qquad V = \mu\, e^{b\phi} \quad ,$$

with $Q = b + b^{-1} = \frac{10-D}{4\alpha'}$, is a self-consistent perturbative string background at low energy. The tachyon background is strongly coupled at high energy:

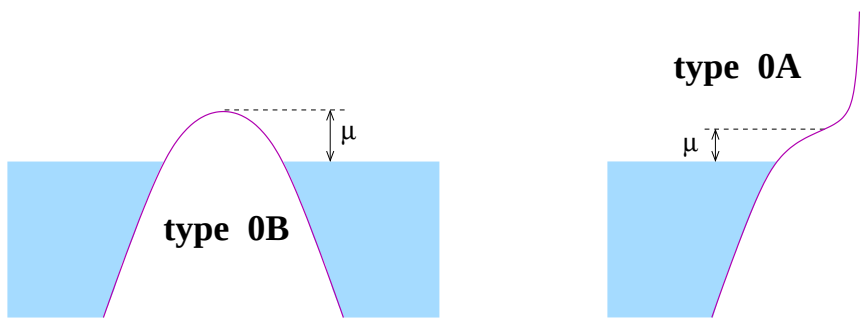


The model has an open string dual that controls the high energy behavior – large N matrix QM on N unstable $D0$-branes (type 0B), or N $D0$-$\overline{D0}$ pairs (type 0A). The open string tachyon condensate on the decayed branes appears to realize the dynamics of closed string worldsheets, neatly bringing together the two avenues of investigation.

The Gauss law of the open string gauge field projects the dynamics onto the $U(N)$ singlet sector of the matrix. The physical degrees of freedom are the matrix eigenvalues, which behave as free fermions in an appropriate potential. The potential $\mathcal{U}(\lambda)$ in the continuum limit is an inverted oscillator $\mathcal{U}(\lambda) = -\frac{\lambda^2}{4\alpha'}$ for type 0B, and

$$\mathcal{U}(\lambda) = -\frac{\lambda^2}{4\alpha'} + \frac{q^2 - 1/4}{2\lambda^2}$$

for type 0A (q is the net $D0$ charge of the background). We focus on the type 0A theory, in order to eliminate subtleties associated to the $2 \longrightarrow 1$ map of λ-space to ϕ-space.



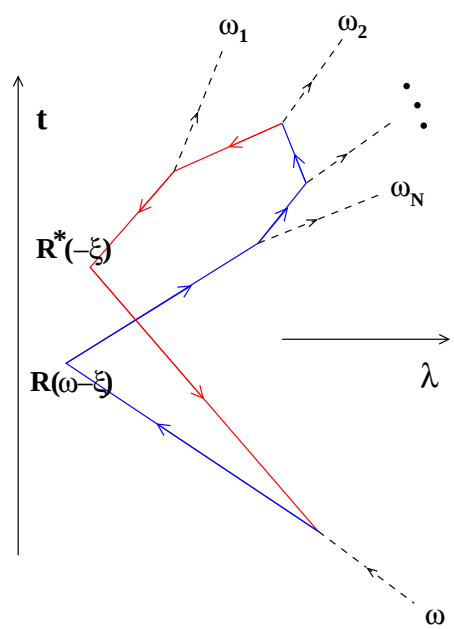
High-energy scattering

What does high-energy scattering look like? The free fermions enable us to compute the S-matrix exactly (*Moore-Plesser-Ramgoolam 9111035*).

The only physical state of the 2D type 0A string (at generic momenta) is the tachyon; in the matrix formulation, the tachyon is a density wave on the Fermi surface

$$V_\omega = e^{i\omega(x\pm\phi)} e^{Q\phi/2} \quad \Leftrightarrow \quad \delta\rho = \delta\psi^\dagger\psi \sim \frac{1}{2\lambda}\, e^{i(x\mp\omega\log|\lambda|)}$$

The S-matrix is evaluated by LSZ reduction of correlations of fermion bilinears. For instance for $1 \longrightarrow N$ scattering



The scattering fermions pick up a phase in reflecting off the potential,

$$\psi(\lambda) \overset{\lambda\to-\infty}{\sim} \frac{1}{\sqrt{\pi\lambda}}\left[e^{-i\lambda^2/2+i\omega\log|\lambda|} + R(\omega)e^{-i\lambda^2/2-i\omega\log|\lambda|}\right]$$

i.e. $R(\omega)$ is the reflection coefficient for the one-particle QM problem in the potential $\mathcal{U}(\lambda)$:

$$R(\omega) = \left|\frac{q^2}{4} - \frac{1}{16} + \mu^2\right|^{-i\omega} \frac{\Gamma(\frac{1}{2}+\frac{q}{2}+i\omega-i\mu)}{\Gamma(\frac{1}{2}+\frac{q}{2}-i\omega+i\mu)}$$

Holes pick up $R^*(-\omega)$. (Note that we measure energy w.r.t. the fermi level $-\mu$.)

There are also "leg-pole" factors in the relative normalization of the worldsheet and matrix model representatives of the asymptotic states

$$V_\omega = \frac{\Gamma(+i\omega)}{\Gamma(-i\omega)}\,\delta\rho(\omega)$$

we will consider quantities such as the number distribution in the final state, for which these factors play no essential role. *Note, however, that these factors are essential for reproducing low-energy gravitational effects, such as the scattering of a tachyon from the gravitational field produced by another tachyon* (*Natsuume-Polchinski 9402156*).

The $1 \longrightarrow N$ amplitude is thus

$$\mathcal{A}_N(\omega|\omega_i) = \sum_{S\subset\{1,\cdots,N\}} (-1)^{|S|} \int_{\omega_S}^{\omega} d\xi\, R(\omega-\xi) R^*(-\xi)$$

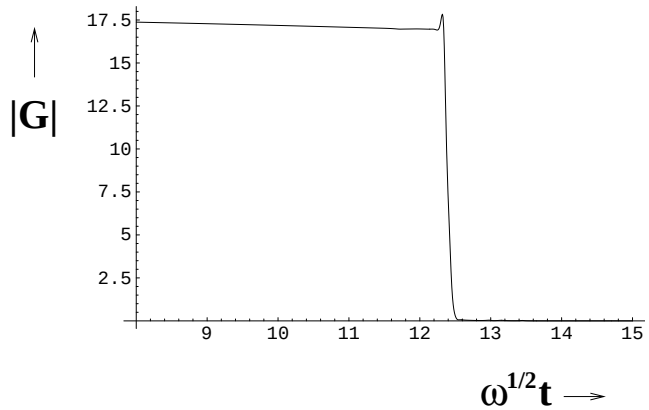
where $\omega_S = \sum_{l=1}^{k} \omega_{j_l}$ and $|S| = k$ for $S = \{j_1, \cdots, j_k\}$. With a few tricks, one can do the sum:

$$\mathcal{A}_N = \left|\tfrac{q^2}{4} - \tfrac{1}{16} + \mu^2\right|^{-i\omega} i^{N-1} \int_{-\infty}^{\infty} \frac{dt}{2t}\, G(t)\, e^{2it\mu} \prod_{j=1}^{N} 2\sin(\omega_j t)$$

where $G(t)$ is the Fourier transform of the reflection amplitude of the particle-hole pair (so t is the time difference in the bounce of the particle vs. that of the hole)

$$G(t) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi}\, e^{it\nu} F\left(\tfrac{\omega-\nu}{2}\right) F\left(\tfrac{\omega+\nu}{2}\right) \quad , \qquad F(\xi) = \frac{\Gamma(\frac{1}{2}+\frac{q}{2}+i\xi)}{\Gamma(\frac{1}{2}+\frac{q}{2}-i\xi)}$$

The function $G(t)$ is approximately constant for $t \lesssim 1/\sqrt{\omega}$, with $G(0) \sim \sqrt{\frac{\omega}{2\pi}}$:



One of the physically interesting quantities we can calculate is the number distribution $\mathcal{P}_N$ of final state tachyons in $1 \longrightarrow N$ scattering, which is defined by

$$\mathcal{P}_N = \frac{1}{N!}\frac{1}{\omega}\int_0^\infty \prod_{i=1}^{N} \frac{d\omega_i}{\omega_i}\,\delta\Big(\sum_i \omega_i - \omega\Big)\Big|\mathcal{A}_N(\omega|\omega_i)\Big|^2 .$$

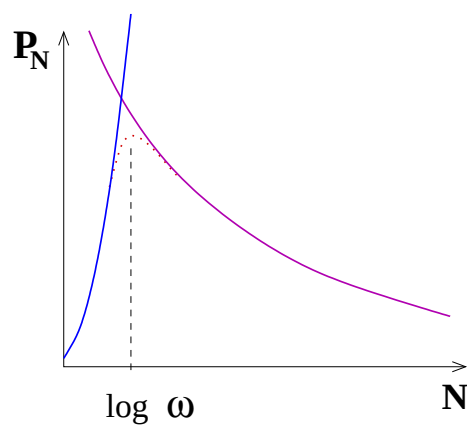
For N of order a few, the integral for $\mathcal{A}_N$ is dominated by the region where G is constant; one can pull $|G(0)|^2$ out of the integrals, and then compute them in closed form with the result (we set $\mu = 0$, in which case $\mathcal{P}_{2k} \equiv 0$)

$$\mathcal{P}_{2k-1} \sim \frac{(2\pi)^{2k-1}(2^{2k}-1)|B_{2k}|}{(2k)!\,\omega} \sim \frac{2^{2k}}{\omega}$$

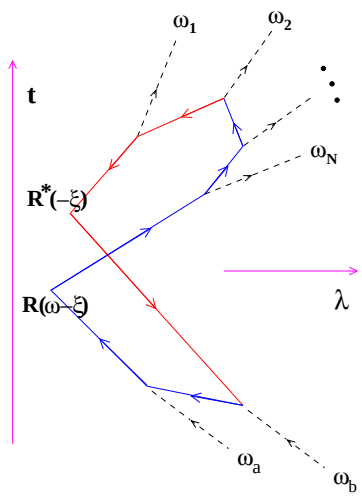
For N large enough that we can treat the final states statistically, we can approximate the integral over the final state energies by saddle point, with $\langle \omega_i \rangle = \omega/N$. The saddle point integral leads to

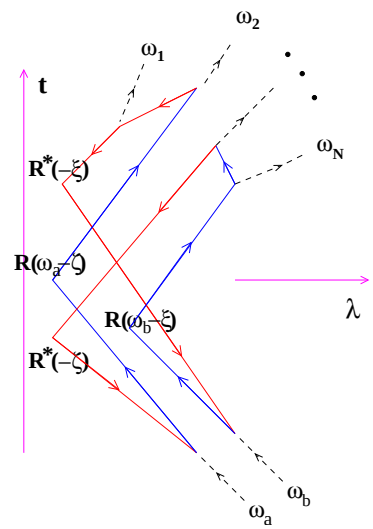
$$\mathcal{P}_N \sim \frac{1}{N!}\frac{1}{\omega}\left(\frac{8}{\sqrt{\pi}}\right)^N$$

To summarize, the number distribution for $1 \longrightarrow N$ scattering is peaked at $N_{\max} \sim \log \omega$:



Similar results apply for $2 \longrightarrow N$ scattering. There are two ring diagrams





Consider for example both incoming particles of the same energy; for $\mu = 0$ and N even

$$\mathcal{A}_{2\to N}\left(\frac{\omega}{2},\frac{\omega}{2}\middle|\,\omega_i\right) = \frac{1}{\pi}\int_0^\infty \frac{dt}{t}\int_0^\infty \frac{ds}{s}\left\{G_\omega(t)\left[\prod_{l=1}^N 2i\sin\omega_l(s+t) - \prod_{l=1}^N 2i\sin\omega_l(s-t)\right]\right.$$

$$\left. - \sum_{A\cup B=\{1,\cdots N\}} G_{\omega_A} * G_{\omega_B}(t)\left[\prod_{l\in A} 2i\sin\omega_l(s+t) - \prod_{l\in A} 2i\sin\omega_l(s-t)\right]\prod_{j\in B}(-2i)\sin\omega_j s\right\}$$

Here $f*g$ is the convolution $f*g(t) = \int_{-\infty}^{\infty} du f(u)g(t-u)$. Qualitatively, this amplitude will have a similar structure to the $1 \longrightarrow N$ amplitude. The $k \longrightarrow N$ distribution is also much the same – for N large enough that statistical methods apply, the probability to find N particles in the final state is a steeply decreasing function of N.

Black holes?

This result contrasts strongly with expectations from the low energy effective theory of dilaton gravity. This theory has a 2D black hole solution

$$\begin{aligned} ds^2 &= d\phi^2 \pm \tanh^2(\tfrac{1}{2}Q\,\phi)\,dx^2 \\ \Phi &= \Phi_0 + \log[\cosh(\tfrac{1}{2}Q\,\phi)]\ , \end{aligned}$$

with $Q^2 = \frac{2}{k-2} = 4$ $(k = 5/2)$ for the 2D type 0 string. This background admits an exact worldsheet description as the $\frac{SL(2,\mathbb{R})}{U(1)}$ gauged WZW model, and so naively is on as solid ground as the tachyon background described by Liouville CFT.[1]

Can one form black holes in 2D string theory? If black holes can form, they should be the generic intermediate states due to their large density of states; they should dominate the behavior of the S-matrix. The thermodynamics of such black holes is

$$S_{\rm BH} \sim \ell_{\rm s} E \quad , \qquad T_{\rm H} \sim \ell_{\rm s}$$

[1] The low-energy theory also has RR charged black hole solutions when $q \neq 0$; it is not known whether there is a corresponding exact worldsheet CFT.

The black hole thus emits its energy E in the form of quanta of string scale energy. The expected number of emitted quanta would be

$$\langle N \rangle \sim \ell_{\rm s} E$$

which is *not* what is found in the exact S-matrix, whose number distribution is peaked at $\langle N \rangle \sim \log E$.

The reason that it is unlikely to find a large number of outgoing particles in the scattering of a small, fixed number of high-energy ingoing particles stems from the basic structure of the ring diagrams. For an initial state of k tachyons, the number of particle-hole pairs in the intermediate fermion state is at most k. This state has a very small overlap with an outgoing state

$$|\mathrm{out}\rangle = \psi^\dagger \psi_1 \, \psi^\dagger \psi_2 \cdots \psi^\dagger \psi_N |\mathrm{vac}\rangle$$

where in normal ordering the product of fermion operators we must select only those with at most k particle and hole creation operators. This is a very small fraction of the totality of excitations that can be created by this operator.

This result suggests that black holes do not form in 2D string theory. In fact, *many aspects of 2D string theory indicate that there is nothing in 2D string theory that could reasonably be called a black hole:*

Entropy: Matrix quantum mechanics in the singlet sector does not have a Hagedorn density of states. In many examples of the gauge/gravity correspondence, the formation of black holes is associated with a deconfinement transition in the gauge theory, in which one begins to access the non-singlet degrees of freedom of the gauge theory ($e.g.$ $AdS_5 \times \mathbb{S}^5/\mathcal{N}=4$ SYM). These degrees of freedom are projected out of the matrix model of 2D string theory.

Conservation laws: The free fermionic character of the matrix model leads to an infinite set of conserved quantities; the time-independent ones are

$$\mathcal{Q}_\ell = \int d\varepsilon \; \varepsilon^\ell \, b^\dagger_\varepsilon b_\varepsilon$$

where $b, b^\dagger$ are fermion creation and annihilation operators. Sen (*0408064*) has argued that the conserved charges of the 2D black hole are $\mathcal{Q}_1 = \omega, \;\; \mathcal{Q}_\ell = 0$.

States with these properties in the free fermion Hilbert space consist of a macroscopic number of very soft tachyon excitations – more or less a coherent state of soft tachyons:

$$\mathcal{Q}_1 = \sum_{i=1}^{N} \pm\omega_i = \omega \quad ; \qquad \mathcal{Q}_\ell = \sum_{i=1}^{N} \pm\omega_i^\ell \sim o(N^{1-\ell}) \ , \quad \ell > 1$$

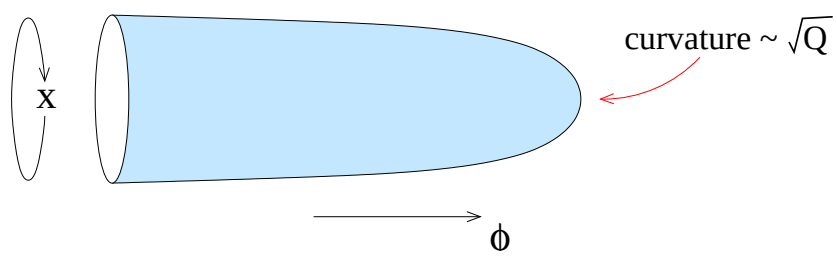
The black hole charges are approached in the limit of large N. Since the phase space of such states is not exponentially large, there is no reason for the evolution to be attracted to such states as intermediate states. Furthermore, the process of soft tachyons making up this state generating a Hawking quantum of energy $m_{\rm s}$ is the time reverse of the process of one energetic tachyon making many soft ones, which we argued above is kinematically suppressed. The outgoing state will not look like Hawking radiation from a 2D black hole.

The worldsheet CFT: The string scale curvature near the horizon of the dilaton black hole geometry

$$\begin{aligned} ds^2 &= d\phi^2 \pm \tanh^2(\tfrac{1}{2}Q\,\phi)\,dx^2 \\ \Phi &= \Phi_0 + \log[\cosh(\tfrac{1}{2}Q\,\phi)]\ , \end{aligned}$$

points to the importance of an underlying exact CFT, the $\frac{SL(2,\mathbb{R})}{U(1)}$ gauged WZW model.

The putative Euclidean black hole geometry is a capped semi-infinite cylinder often referred to as the "cigar":



The background can be thought of as the nonlinear completion of the linearized deformation $e^{Q\phi}\partial X\bar{\partial}X$ of the metric away from flat spacetime. A shift in ϕ makes $e^{-2\Phi_0}$ the coupling in front of the asymptotic graviton; as in higher dimensions, this coefficient of the leading asymptotic deformation (of the time component G_{00} of the metric away from flat spacetime) is the mass of the black hole $\mu_{\rm bh} = e^{-2\Phi_0}$.

A great deal is known about this CFT. There is a conformal bootstrap, analogous to that of Liouville theory. In Liouville theory, one uses the two degenerate operators

$$V_{-b/2} = e^{-b\phi/2} \quad , \qquad V_{-1/2b} = e^{-\phi/2b}$$

to derive constraints on correlators which fix them uniquely. The analogous operators in the coset are the degenerate primary operators Φ_j of $SL(2,\mathbb{R})$ current algebra, having spin $j = -\frac{3}{2}$ and $j = -\frac{k}{2}$.

The coupling constant of Liouville theory is the parameter b. The theory is self-dual under $b \longrightarrow 1/b$, which preserves the slope $Q = b + 1/b$ of the linear dilaton, and exchanges the two degenerate operators.

The Euclidean $\frac{SL(2,\mathbb{R})}{U(1)}$ gauged WZW model also has a strong/weak coupling duality, but it is not self-dual. Rather, the dual is the Sine-Liouville theory

$$\mathcal{S}_{\rm SL} = \frac{1}{4\pi}\int d^2z d^2\theta \Big[(D\phi\bar{D}\phi + DX\bar{D}X) + QR^{(2)}\phi + \mu_{\rm sl}\cos R[X_l - X_r]\, e^{\frac{1}{Q}\phi}\Big]$$

where again $Q^2 = \frac{2}{k-2}$; X is compactified on a circle of radius $R = 2/Q$, and $X_l - X_r$ is the axial component of X. The conformal bootstrap for this theory yields the same correlation functions as the coset model, and determines a relation $\mu_{\rm sl} = \mu_{\rm sl}(\mu_{\rm bh})$.

In a sense, the duality exchanges the roles of the asymptotic graviton interaction $\mu_{\rm bh}\, e^{Q\phi}\partial X\bar{\partial}X$, and the Sine-Liouville interaction $\mu_{\rm sl} e^{\frac{1}{Q}\phi}\cos R(X_l - X_r)$. The metric deformation is the dominant asymptotic (at $\phi \to -\infty$) for $Q \ll 1$, while the Sine-Liouville coupling is dominant for $Q \gg 1$. Since $Q = 2$ for the type 0 string, it appears that the Sine-Liouville description is somewhat more appropriate for this 2D string background.

Note that the Sine-Liouville potential acts as a generating function for vortices in the worldsheet partition function. The $\frac{SL(2,\mathbb{R})}{U(1)}$ coset/Sine-Liouville equivalence leads to a natural candidate for a matrix model equivalent to the Euclidean "black hole" – simply turn off the Liouville potential and turn on a condensate of vortices (winding tachyon) in the compactified Euclidean theory (*Kazakov-Kostov-Kutasov 0101011*). But again, the matrix description of the background has a closer affinity to a tachyon condensate than a Euclidean black hole.

In higher dimensions, when the curvature of a black hole reaches string scale, it undergoes a (correspondence) phase transition to a gas of strings. The apparent dominance of the Sine-Liouville coupling is an indication that the "black hole" of 2D string theory is actually on this other side of the correspondence point, where it is better thought of as a gas or condensate of strings.

If one considers the family of noncritical string backgrounds $\frac{SL(2,\mathbb{R})}{U(1)} \times \mathrm{CFT}_{\hat{c}}$, one can vary Q by varying $\hat{c}$, $Q^2 = 4 - \frac{1}{2}\hat{c}$. The Hagedorn temperature of a perturbative string gas is

$$\beta_{\mathrm{H}}^{\mathrm{pert}} = 2\pi\sqrt{\hat{c}\alpha'/4}$$

while the Hagedorn temperature of the black hole geometry is given by the asymptotic radius of the cigar

$$\beta_{\mathrm{H}}^{\mathrm{cigar}} = 2\pi\sqrt{2\alpha'/Q^2}$$

The two are equal when $Q^2 = 2$, (k=3), which is precisely the point at which the Sine-Liouville interaction starts to dominate. Thus the crossover between dominance of the cigar metric and dominance of the winding tachyon condensate appears to be related to the correspondence transition. In this regard, recall that the Hagedorn transition of perturbative strings can be described as the condensation of precisely this winding tachyon (*Atick-Witten 1988*).

Conclusions

- The D-particles of 2D string theory dominate high energy scattering. Essentially, a high energy incoming tachyon becomes a high energy D-particle, together with the hole it leaves behind; radiation from the D-particle is strongly suppressed.
- The nonperturbative formulation allows an exact formulation of the S-matrix. Estimates of the resulting integrals show that the number distribution is peaked around $N \sim \log \omega$ outgoing particles.
- 2D dilaton black holes would give $N \sim \omega$ outgoing particles.
- The 2D black hole is in a stringy regime. A variety of points of view suggest that stringy effects dominate to the extent that black holes do not form.