

UNITARITY AND BLACK HOLES

on

HAWKING RADIATION FOR DUMMIES

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based on

9907001 (MP + F. WILCZEK)

0402166 (M.P.)

0204107 (generalization to de Sitter Space)

0405160 (gravity-prize-winning essay)

See also

gr-qc/9408003 (Kraus + Wilczek)

9610045 (Kraus + Kestin-Vakkuri)

Summary

1. Hawking radiation is due to tunneling
2. Energy conservation supplies the barrier
3. The emission spectrum is not precisely thermal

$$\Gamma \sim e^{\Delta S} \approx e^{-\beta E}$$

hinting at underlying unitarity

Painlevé Co-ordinates

$$t = t_s + 2\sqrt{2Mr} + 2M \ln \frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}}$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + 2\sqrt{\frac{2M}{r}} dt dr + dr^2 + r^2 d\Omega^2$$

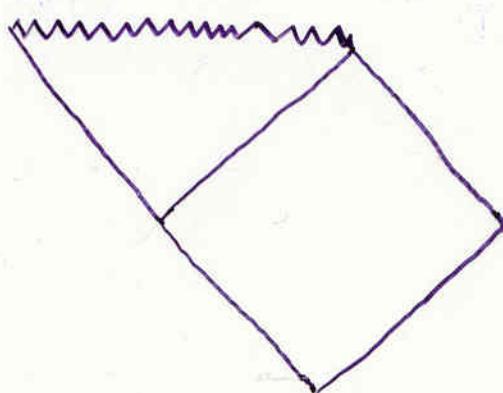
non-singular at the horizon

stationary

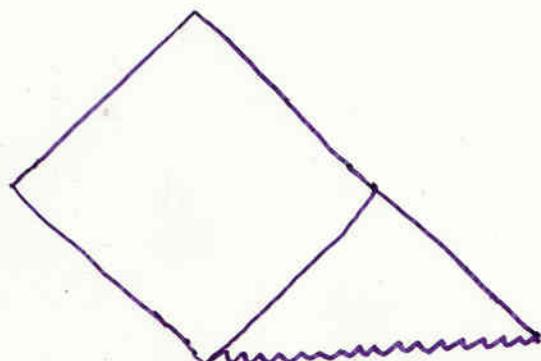
non-static

constant-time sections are flat

~Unruh vacuum



$t \rightarrow -t$



Equations Of Motion

$$\dot{r} = \frac{dr}{dt} = \pm 1 - \sqrt{\frac{2M}{r}}$$

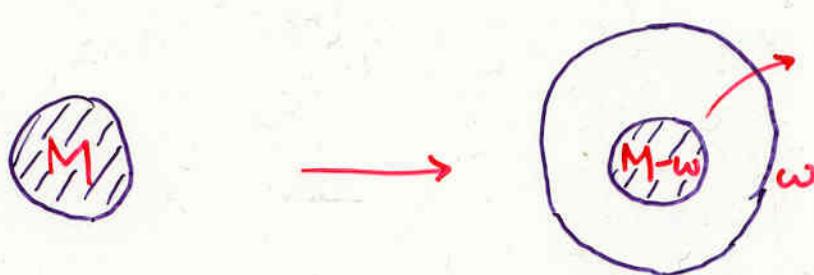
[outgoing] [ingoing]

Effect of self-gravitation on a shell (s-wave)

black hole mass fixed,
total mass variable: $M \rightarrow M + \omega$

KRAUS + WILCZEK

black hole mass variable,
total mass fixed: $M \rightarrow M - \omega$



Outgoing, massless, self-gravitating shell:

$$\dot{r} = +1 - \sqrt{\frac{2(M-\omega)}{r}}$$

Tunneling Rate

$$\Gamma \sim e^{-2\text{Im}S} \quad (\text{WKB})$$

$$\begin{aligned}\text{Im } S &= \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} p_r dr \\ &= \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \int_0^{p_r} dp'_r dr \\ &= \text{Im} \int_0^{+\omega} \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{dr}{1 - \sqrt{\frac{2(M-\omega')}{r}}} (-d\omega') \\ &= +4\pi\omega(M - \frac{\omega}{2})\end{aligned}$$

$$\therefore \Gamma \sim e^{-8\pi M \omega (1 - \frac{\omega}{2M})}$$

$$\Theta((\frac{\omega}{M})^0): \Gamma \sim e^{-\beta\omega}, \beta = 8\pi M - \text{Hawking radiation}$$

$$g(\omega) = \frac{1}{2\pi} \frac{|T(\omega)|^2}{e^{+8\pi M \omega} - 1} \quad \text{- Planckian spectral flux}$$

$$\Theta((\frac{\omega}{M})'): \Gamma \sim e^{+\Delta S_{\text{B-H}}}$$

non-thermal spectrum

Quick Check

$$\Delta\omega = \omega/N$$

For N large, Hawking's thermal formula should hold:

$$\Gamma \sim e^{-\beta \Delta\omega}$$

Sum up infinitesimals:

$$\Gamma \sim e^{-8\pi M \Delta\omega} e^{-8\pi (N-\Delta\omega) \Delta\omega} \dots e^{-8\pi (M-(N-1)\Delta\omega) \Delta\omega}$$

$$\stackrel{N \rightarrow \infty}{=} e^{-8\pi M \left(\omega - \frac{\omega^2}{2M} \right)}$$

$$= e^{+\Delta S_{B-H}}$$

Where Is The Barrier?

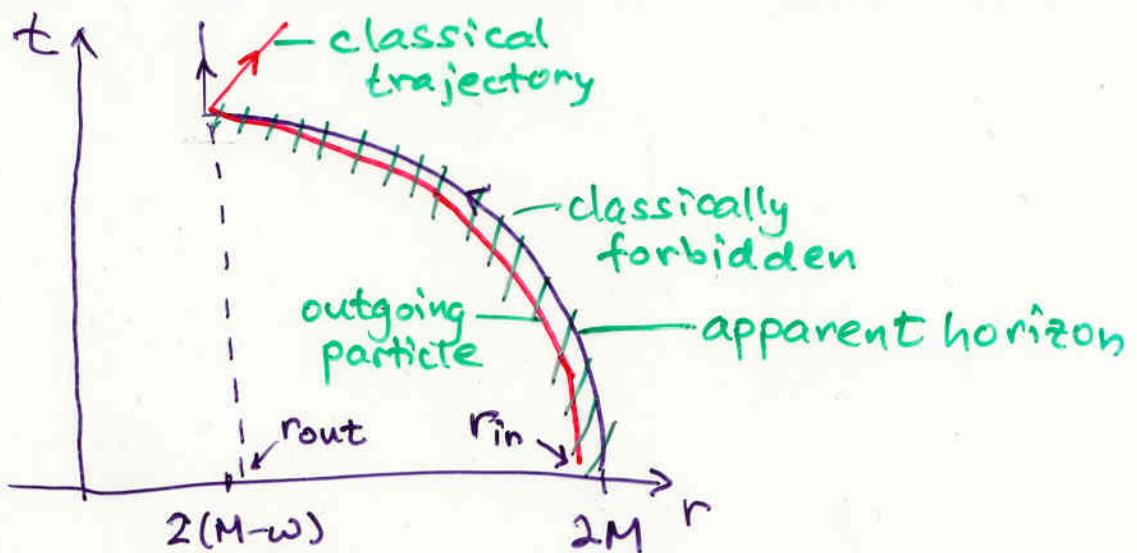
$$\text{Im } S = \text{Im} \int_{r_{in}}^{r_{out}} \int_M^{M-\omega} \frac{dM'}{1 - \sqrt{\frac{2M}{r}}} dr$$

$$= \text{Im} i \int_{r_{in}}^{r_{out}} (-\pi r) dr$$

$$\therefore r_{in} = 2M$$

$$\therefore r_{out} = 2(M-\omega)$$

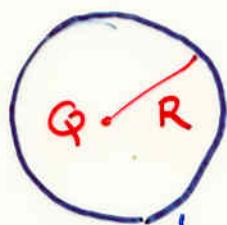
Over the course of the classically forbidden trajectory, the outgoing particle travels radially inward with the apparent horizon to appear at its final location at $r = 2(M-\omega)$.



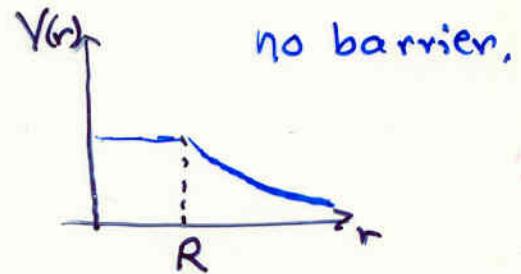
Note role of energy conservation!

Electrostatic Analogy

Consider:



$$V(r) = \frac{qQ}{r}$$



↳ charged, conducting shell /sphere

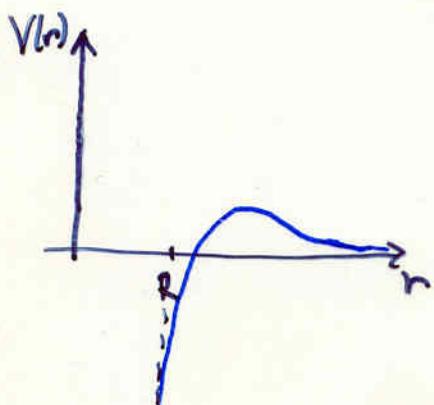
But charge must be conserved:

$$Q \rightarrow Q - q.$$



↳ image charge

$$V(r) \rightarrow q \left(\frac{Q-q}{r} - \frac{qR}{r^2 - R^2} \right)$$



charge conservation
induces a barrier!

This phenomenon is known as Coulomb blockade
(important in single-electron microelectronics).

Exact Result

$$\Gamma \sim e^{+\Delta S_{B-H}} \quad \text{nonthermal spectrum!}$$

But is this correct? Check...

- Probability (Boltzmann):

$$P(\text{shell with energy } \omega) = \frac{\# \text{ of states of b.h. with mass } M-\omega}{\text{total } \# \text{ of states}}$$

e.g. $\omega = M$:

$$P(\omega = M) = \frac{1}{e^{+S_i}} = e^{-4\pi M^2} = e^{+\Delta S_{B-H}}$$

cf. naive result: $P = e^{-8\pi M\omega} = e^{-8\pi M^2}$

- Microscopics:

$$\text{Rate} = |\text{amplitude}|^2 \times \text{phase space factor}$$

Phase space factor: sum over final states,
average over initial states

$$\therefore \text{phase space factor} = \frac{e^{+S_f}}{e^{+S_i}} = e^{+\Delta S_{B-H}}$$

AdS Black Holes

$$ds^2 = -F(r) dt_{HP}^2 + \frac{dr^2}{F(r)} + r^2 d\Omega_{d-1}^2$$

$$F(r) = 1 - \frac{\mu}{r^{d-2}} + \frac{r^2}{L^2}, \quad \mu = \frac{16\pi G_{d+1}}{(d-1)\Omega_{d-1}} M$$

$$t = t_{HP} + f(r), \quad f'(r) = -\frac{1}{r^{\frac{d-2}{2}} F(r)} \sqrt{\frac{\mu}{1+r^2/L^2}}$$

$$\therefore ds^2 = -F(r) dt^2 + \frac{2}{r^{\frac{d-2}{2}}} \sqrt{\frac{\mu}{1+r^2/L^2}} dt dr + \frac{dr^2}{1+r^2/L^2} + r^2 d\Omega_{d-1}^2$$

\sim Painlevé - Hawking - Page line element

$$\dot{r} = -\frac{\sqrt{\mu}}{r^{\frac{d-2}{2}}} \sqrt{1+r^2/L^2} \pm (1+r^2/L^2) \quad \text{- equation of motion}$$

$$M \rightarrow M - \omega \quad \text{- self-gravitation}$$

$$\text{Im } S = \text{Im} \int_0^\omega d\omega' \int dr \frac{1 + \sqrt{\frac{\mu}{r^{d-2}} \frac{1}{1+r^2/L^2}}}{F(r)}$$

$$= \pi \frac{(d-1) \Omega_{d-1}}{16\pi G_{d+1}} \int d\mu' 2 r_H^{d-2} \frac{dr_H}{d\mu}$$

$$= \frac{1}{2} \Delta S_{BH}$$

Tunneling From A Charged Hole

$$t = t_r + 2\sqrt{2Mr - Q^2} + M \ln \left(\frac{r - \sqrt{2Mr - Q^2}}{r + \sqrt{2Mr - Q^2}} \right) + \sqrt{Q^2 + M^2} \operatorname{arctanh} \left(\frac{\sqrt{M^2 - Q^2} \sqrt{2Mr - Q^2}}{Mr} \right)$$

$$ds^2 = - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + 2\sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}} dt dr + dr^2 + r^2 d\Omega^2$$

Equation of motion of an unchanged, outgoing, massless, self-gravitating shell:

$$\dot{r} = \pm \sqrt{\frac{2(M-\omega)}{r} - \frac{Q^2}{r^2}}$$

$$\therefore \text{Im } S = \int_0^{+\infty} \int_{r_{\min}}^{r_{\text{out}}} \frac{dr}{\sqrt{r - \sqrt{\frac{2(M-\omega)}{r}} - \frac{Q^2}{r^2}}} (-d\omega')$$

$$(u = \sqrt{2(M-\omega') r - Q^2}, \omega' \rightarrow \omega' - i\epsilon)$$

$$= 2\pi \left(2\omega(M-\omega) - (M-\omega) \sqrt{(M-\omega)^2 - Q^2} + M \sqrt{M^2 - Q^2} \right)$$

The term $\sqrt{(M-\omega)^2 - Q^2}$ implies the third law of black hole thermodynamics is manifest.

Tunneling rate:

$$\Gamma \sim e^{-\beta \omega}, \beta = \frac{2\pi (M + \sqrt{M^2 - Q^2})^2}{\sqrt{M^2 - Q^2}}$$

$$\Gamma = A e^{+\Delta S_{B-H}}$$

Stretched Horizon Picture

Action for stretched horizon:

$$S = \frac{1}{8\pi} \int d^3x \sqrt{h} K$$

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gr-qc/9712077

At fixed Schwarzschild radius,

$$\sqrt{h} = \sqrt{1 - \frac{2M}{r}} r^2 \sin\theta \xrightarrow{r \rightarrow 2M} 0$$

$$K = \frac{1}{\sqrt{1 - \frac{2M}{r}}} \left(\frac{M}{r^2} + \frac{2}{r} (1 - \frac{2M}{r}) \right) \xrightarrow{r \rightarrow 2M} \infty$$

Complexify time:

$$t \rightarrow -i\tau, \int d\tau = \beta = 8\pi M, d\tau = 8\pi dM$$

Compute action:

$$\begin{aligned} \text{Im } S &= -\frac{1}{8\pi} \int 8\pi dM \cdot 4\pi r^2 \left(\frac{M}{r^2} + \frac{2}{r} (1 - \frac{2M}{r}) \right) \\ &= -4\pi \int M dM \end{aligned}$$

Probability for spontaneous emission by stretched horizon:

$$\Gamma \sim e^{-2\text{Im } S} = e^{\Delta S_{B-H}}$$

Correlations?

Probability of emitting particle with energy E_1 , followed by particle with energy E_2

$$= \exp \left(-8\pi \left(E_1 \left(M - \frac{E_1}{2} \right) + E_2 \left(M - E_1 - \frac{E_2}{2} \right) \right) \right)$$

$$= \exp \left(-8\pi \left((E_1 + E_2) \left(M - \frac{E_1 + E_2}{2} \right) \right) \right)$$

= prob. of emitting a single particle with energy $E_1 + E_2$.

Horizon doesn't shrink instantaneously ...

... so there could be short-time

temporal correlations in Hawking radiation.