The Gregory-Laflamme instability for the D2-D0 bound state

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Correlated stability conjecture (CSC) *[Gubser-Mitra '00]*:

- A Gregory-Laflamme (GL) instability arises precisely when there's a thermodynamic instability.

What's a thermodynamic instability?

- Say $S = S(E, Q)$.
- Form $\mathbf{H} = \begin{pmatrix} \partial^2 S/\partial M^2 & \partial^2 S/\partial M \partial Q \\ \partial^2 S/\partial M \partial Q & \partial^2 S/\partial Q^2 \end{pmatrix}$.
- If $\mathbf{H}$ has a negative eigenvalue, there's an instability.
- A negative eigenvalue means you can gain S by redistributing E and/or Q non-uniformly.
- GL instability does precisely this.

Two caveats:

- You must *be able* to redistribute Q. Else exclude it from Hessian.
  - Example: N coincident D1's. Q_{D1} can't be redistributed along the D1's.
  - Example: array of D0's, N per cm^2. Q_{D0} *can* be redistributed, making array non-uniform.
- Finite size effects could prevent a GL instability
  - Example: Black string on an S^1 of radius R has a GL instability only if $R > r_H$ (up to an $O(1)$ factor).
  - Example: Black hole in 4-d has a finite size horizon: stable despite thermodynamic instability, $C < 0$.

Why does it work?

- GL noted that $S_{\text{blackhole}} > S_{\text{blackstring}}$ for large mass. Saying $\mathbf{H} \geq 0$ is a local version of this.
- GL instability is an *infrared effect*, so it makes sense for thermodynamics to dominate it.

In practice...

- Dispersion curve has $\omega^2(k) < 0$ for $k < k_*$ (the unstable modes).
- Simplest to look for the *static* perturbation at $k = k_*$: $\text{fields} \sim \cos kx$ for some x along the brane.

Understanding of GL seems still rather primitive! (Numerics always seems required).

1. Endpoint of evolution of unstable horizons not known.

2. Existence of instability checked only for simple situations.

Point 2 is where CSC helps. There's even a proof of CSC if there are no Q's *[Reall '02]*.

Let's apply CSC to D2-D0 bound state and then check the results with numerics.

This is the simplest case where you can't guess the right answer in 5 minutes.

Some intuitions:

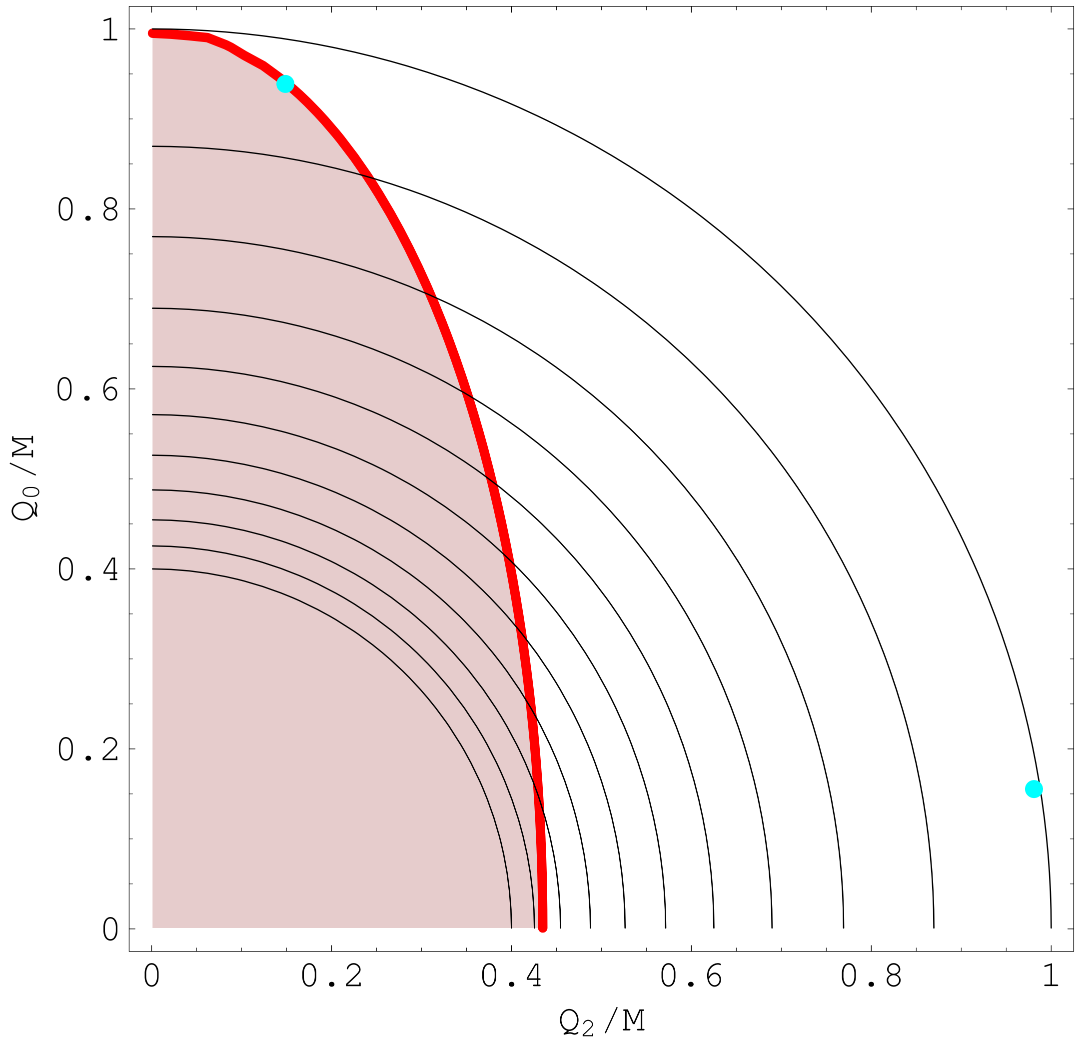
- Highly non-extremal D2-D0 has a GL instability: charges don't matter.
- For extremal D2-D0, $\tau = \sqrt{Q_2^2 + Q_0^2}$, a convex function of Q_0. So making Q_0 non-uniform increases $\int \tau$—not favored.
- If $Q_0 \gg Q_2$, then D2 doesn't matter. Continuous array of D0's should have a GL instability *[Aharony et al '04]*.
- If $Q_0 \ll Q_2$, GL instability appears finitely far from extremality *[Gubser-Ozakin '03]*.

The answer:

CSC predicts GL instability in shaded region only.

Numerics performed for the points indicated agree with this.

Next 8 slides show how I got this.



$$S_{IIA} = \frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{G}\left[e^{-2\phi}\left(R + 4(\partial\phi)^2 - \frac{1}{2}H_3^2\right) - \frac{1}{2}F_2^2 - \frac{1}{2}\tilde{F}_4^2\right]$$

$$\text{with } \tilde{F}_4 = F_4 + A_1 \wedge H_3 \tag{1}$$

$$ds^2_{str} = H^{-1/2}(-hdt^2 + D(dx_1^2 + dx_2^2)) + H^{1/2}\left(\frac{1}{h}dr^2 + r^2 d\Omega_6^2\right)$$

$$H = 1 + \frac{r_0^5 \sinh^2\alpha}{r^5} \qquad D = \frac{1}{H^{-1}\sin^2\theta + \cos^2\theta} \qquad h = 1 - \frac{r_0^5}{r^5}$$

$$A_1 = \coth\alpha\sin\theta\left(1 - \frac{1}{H}\right)dt$$

$$A_3 = \coth\alpha\sec\theta\left(1 - \frac{D}{H}\right)dt \wedge dx^1 \wedge dx^2$$

$$e^{2\phi} = H^{1/2}D \qquad B_2 = \tan\theta\left(1 - \frac{D}{H}\right)dx^1 \wedge dx^2$$

Thermodynamics:

$$
\begin{aligned}
&M = \frac{V_2 \Omega_6}{2\kappa^2} r_0^5 (6 + 5\sinh^2\alpha) \\
&\quad T = \frac{5}{4\pi r_0 \cosh\alpha} \qquad S = \frac{2\pi V_2 \Omega_6}{\kappa^2} r_0^6 \cosh\alpha \\
&\quad \mu_2 = \mu\cos\theta \quad Q_2 = Q\cos\theta \quad \mu_0 = \mu\sin\theta \quad Q_0 = Q\sin\theta \\
&\quad \mu = \tanh\alpha \qquad Q = \frac{5V_2\Omega_6}{2\kappa^2} r_0^5 \sinh\alpha\cosh\alpha
\end{aligned}
\tag{2}
$$

- Want $S = S(M, Q_0, Q_2)$.
- Have S, M, Q_0, Q_2 in terms of r_0, α, θ.
- Use $\left(\frac{\partial f}{\partial y_i}\right)_{y_j} = \frac{|\partial(y_1, \ldots, \hat{y}_i, f, \ldots, y_n)/\partial(x_1, \ldots, x_n)|}{|\partial(y_1, \ldots, y_n)/\partial(x_1, \ldots, x_n)|}$.

Boundary of stability (the red line) is where $\det \mathbf{H}_{2\times 2} = 0$.
2×2 means we exclude Q_2. Get

$$\operatorname{csch} \alpha = \sqrt{3} \cos \theta \,. \tag{3}$$

The plot was in terms of

$$\begin{aligned} Q_0/M &= 5 \sin \theta \sinh 2\alpha / (7 + 4 \cosh 2\alpha) \\ Q_2/M &= 5 \cos \theta \sinh 2\alpha / (7 + 4 \cosh 2\alpha) \end{aligned} \tag{4}$$

with lines of constant α shown.

Reall's proof of CSC is for no Q's. Hopefully extendable. Following analysis may help pave the way.

We seek a *static, non-uniform solution.*
So KK reduce to only the $(x^1, r) = (x^\mu)$ directions:

$$\begin{aligned} ds^2_{str} &= G_{\mu\nu} dx^\mu dx^\nu - e^{2\varphi_1} dt^2 + e^{2\varphi_2} dx_2^2 + e^{2\varphi_3} d\Omega_6^2 \\ A_1 &= a_1 + a_0 dx^2 + \tilde{a}_0 dt \quad \ldots \end{aligned} \tag{5}$$

Gauge fields like a_1 have no dynamics, but they do impose constraints. Suppose for example

$$\begin{aligned} S[a_1, \ldots] &= \int \mathcal{L} \\ \mathcal{L} &= \int \frac{1}{2} e^{\varphi} (f_2 + q_2) \wedge *(f_2 + q_2) + q_0 f_2 \\ \pi_a &\equiv *(f_2 + q_2) + q_0 = const \\ \hat{S}[\pi_a, \ldots] &= \int (\mathcal{L} - \pi_a f_2) \; . \end{aligned} \tag{6}$$

$\hat{S}$ gives correct dynamics for fields other than a_1.

Some ghastly mess results for dynamics of $G_{\mu\nu}$, φ_1, φ_2, φ_3, $\tilde{a}_0$, and the 2-d dilaton

$$\Phi = \phi - \frac{1}{2}\left(\varphi_1 + \varphi_2 + 6\varphi_3\right) \; . \tag{7}$$

Schematically:

$$G^{-1/2}L = e^{-2\Phi}\left(R + 4(\partial\Phi)^2\right) - \frac{1}{2}\mathcal{G}_{ab}(\varphi)\partial\varphi^a\partial\varphi^b - V(\varphi)\, . \tag{8}$$

Write $G_{\mu\nu} = e^{2\sigma}G^{\rm background}_{\mu\nu}$: a conformal gauge.

Then perturbations are expressed in terms of six scalars:

$$(\varphi^a) = (\sigma,\ \Phi,\ \varphi_1,\ \varphi_2,\ \varphi_3,\ \tilde{a}_0) \tag{9}$$

(a few others, like a_0, decouple). Ansatz is

$$\varphi^a = \varphi^a_{\rm background}(r) + \delta\varphi^a(r)\cos kx^1\, . \tag{10}$$

$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R = 0$ imposes two 1st order gauge conditions on $\delta\varphi^a(r)$, plus the eom for $\delta\sigma(r)$ (2nd order).

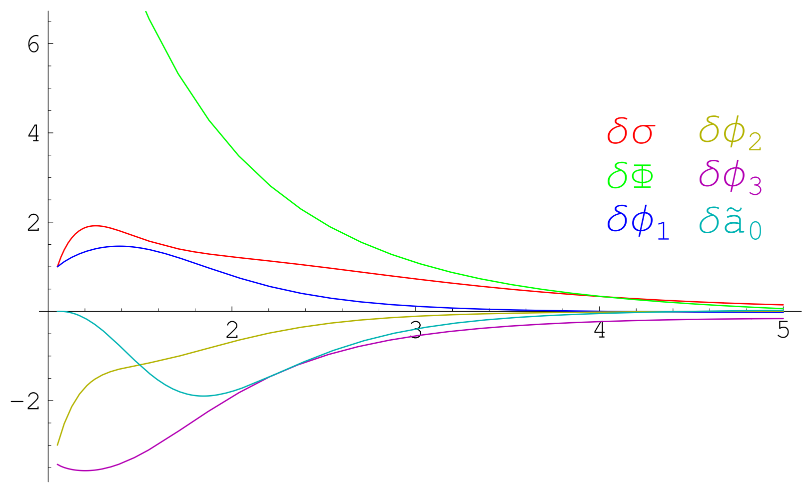
So for 6 functions $\delta\varphi^a(r)$, we have 10 integration constants: $6 \times 2 - 2 = 10$. k is an 11^{th} parameter.

Fix these 11 parameters by

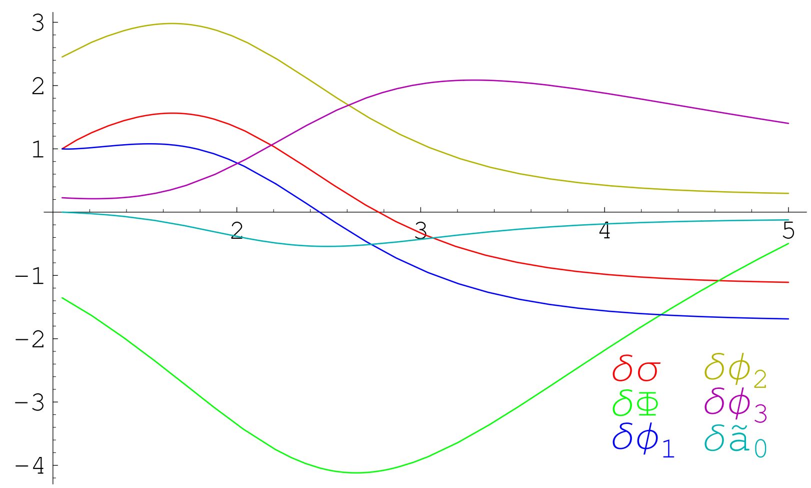
- Normalizability at ∞.
  - Gives 6 conditions: think $\varphi(r_f) = 0$ for some big r_f.
- Regularity at the horizon.
  - Most naturally analyzed in Kruskal coords—haven't done it.
  - $r = r_0$ is a regular singular point of radial eoms. Allows a rough-and-ready analysis, dropping subdominant terms in eoms to obtain bc's.
  - Gives 5 conditions, including $\delta A_t = 0$ at horizon.

Upshot: solve (mostly) 2nd order diffEQ's from $r = r_0$ to $r = \infty$, subject to 5 bc's at $r = r_0$ and 6 at $r = \infty$.

Suitable for a shooting algorithm: randomly fix 6 remaining parameters at $r = r_0$, integrate, check bc's at $r = \infty$.



GL is present



GL not present

Numerics is prelimary—why?

- Analysis of bc's at $r = r_0$ was loose.
- Setting six numerically computed functions of six variables to 0 is not trivial.
- "Whiplash" effect—might find $\varphi^a(r_f) = 0$ as a result of competition between large solns.
  - Better to minimize $\displaystyle\int_{r_f-\delta r}^{r_f+\delta r} dr \sum_a |\varphi^a(r)|^2$.

Non-commutative field theory at finite temperature:

A famous fact [Maldacena-Russo '99]: thermodynamics at leading order in large N is unaffected by non-commutativity.

- Obvious: large $N \leftrightarrow$ planar diagrams
- Planar vacuum diagrams escape non-commutative phases.
- True in supergravity: M, S, T are θ-independent.

But there's a catch: it was tacitly assumed that $[\tilde{x}^1, \tilde{x}^2] = i\vartheta = const$.

Roughly, $\vartheta \propto Q_0$, so if Q_0 gets non-uniform, thermodynamic behavior is vastly different from the $\vartheta = 0$ case.

This happens if M/Q is big enough. Where is the transition to GL instability?

It happens very close to extremality, where $C > 0$.

- So for $0 \leq T < T_c$, bound state is stable.
- And for $T > T_c$, GL instability arises.
- $T = T_c$ is on the red curve, $\det \mathbf{H} = 0$, i.e. $\operatorname{csch} \alpha = \sqrt{3} \cos \theta$.

Unravel various definitions to find:

- $\vartheta = 2\pi\alpha' \tan \theta \gg \alpha'$.
- Tension at extremality from D2's is

$$\frac{Q_2}{V_2} = \frac{5\Omega_6}{2\kappa^2} r_0^5 \sinh \alpha \cosh \alpha \cos \theta = N_2 \tau_{D2} = \frac{N_2}{g_s 4\pi^2 \alpha'^{3/2}} \,. \quad (11)$$

Find $\sqrt{\vartheta} T_c = \frac{\#}{(g_s N_2)^{1/5}} \left(\frac{2\pi\alpha'}{\vartheta} \right)^{3/10} \left[1 + O(\alpha'^2/\vartheta^2) \right]$.

Note $\sqrt{\vartheta} T \to 0$ in the NCFT limit where $\alpha' \ll \vartheta$.

So NCFT description of D2-D0 is strictly a zero-temperature statement. For $T > 0$, $\vartheta = \vartheta(t, \vec{x})$.

Recall the seemingly innocuous rescaling of coords in NCFT:

- $E_{\alpha\beta} = g_{\alpha\beta} + B_{\alpha\beta}$ enters DBI action: $S \sim \int d^3\xi \sqrt{\det E_{\alpha\beta}}$.
- $E^{\alpha\beta} = G^{\alpha\beta} + \dfrac{\vartheta^{\alpha\beta}}{2\pi\alpha'}$, and $G_{\alpha\beta}$ is the open string metric.
- $(\tilde{t}, \tilde{x}^1, \tilde{x}^2) = (t, x^1/\cos\theta, x^2/\cos\theta)$ are chosen so that $G_{\tilde{\alpha}\tilde{\beta}} = \eta_{\alpha\beta}$.

Massless open strings propagate on null geodesics of $G_{\alpha\beta}$, which is to say at $v = \cos\theta$ in the closed string metric $g_{\alpha\beta}$.

$v \ll 1$: the bound state is soft and squishy. Extreme limit of only D0's has no rigidity at all. That's why GL comes in.

Some caveats:

- I'm relying on CSC: numerics only two points (so far).
- I'm using supergravity approximation, so it's conceivable that some $T > 0$ giving a highly curved horizon avoids GL instability.
- CSC as it stands doesn't tell us k_* or the typical $|\omega|$ of the unstable modes.
- Closed strings decouple, right? Probably so, but there is an order of limits issue in discussing thermodynamics.

Conclusions:

- Presence of a Gregory-Laflamme instability is dictated by (local) thermodynamic considerations: $\det \mathbf{H} < 0$.
- Boundary of stability can be arbitrarily close to extremality.
- Non-commutative field theory is a good description of D2-D0 only at $T = 0$ (or possibly T below what SUGRA can see).
- Several types of "charge" may be redistributed: Q_{D0}, angular momenta, etc.
- Near $\det \mathbf{H} = 0$, evolution of GL probably slides us along the boundary of stability. May be easier to find the endpoint.
- Many other systems to study (work in progress with J. Friess).

References

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