Detecting a Stochastic Background of Gravitational Radiation - Background Information


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1 Stochastic gravitational wave backgrounds

Here we briefly describe the standard optimally-filtered cross-correlation technique used to search for a stochastic background of gravitational radiation. Readers interested in more details should consult the original papers [1, 2, 3] or longer review articles (e.g., [4, 5, 6]) for a more in-depth discussion.

1.1 Spectrum

A stochastic background of gravitational radiation is a random gravitational wave signal produced by a large number of weak, independent, unresolved gravitational wave sources. Its spectral properties are described by the dimensionless quantity

$$\Omega_{\text{gw}}(f) := \frac{1}{\rho_{\text{critical}}} \frac{d\rho_{\text{gw}}}{d \ln f},$$

which is the ratio of the energy density in gravitational waves contained in a bandwidth $\Delta f = f$ to the total energy density required (today) to close the universe:

$$\rho_{\text{critical}} = \frac{3c^2 H_0^2}{8\pi G}.$$

$H_0$ is the Hubble expansion rate (today):

$$H_0 = h_{100} \cdot 100 \ \frac{\text{km}}{\text{sec} \cdot \text{Mpc}} \approx 3.24 \times 10^{-18} h_{100} \ \frac{1}{\text{sec}},$$

and $h_{100}$ is a dimensionless factor, included to account for the different values of $H_0$ that are quoted in the literature. Note that $\Omega_{\text{gw}}(f) h_{100}^2$ is independent of the actual Hubble expansion rate, and for this reason we will often focus attention on this quantity, rather than $\Omega_{\text{gw}}(f)$ alone. In addition, $\Omega_{\text{gw}}(f)$ is related to the one-sided power spectral density $S_{\text{gw}}(f)$ via:

$$S_{\text{gw}}(f) = \frac{3H_0^2}{10\pi^2} f^{-3} \Omega_{\text{gw}}(f).$$

Thus, for a stochastic gravitational wave background with $\Omega_{\text{gw}}(f) = \text{const}$, the power in gravitational waves falls off like $1/f^3$.

1.2 Statistical assumptions

The spectrum $\Omega_{\text{gw}}(f)$ completely specifies the statistical properties of a stochastic background of gravitational radiation provided we make enough additional assumptions. Here, we assume that the stochastic background is: (i) isotropic, (ii) unpolarized, (iii) stationary, and (iv) Gaussian. Anisotropic or non-Gaussian backgrounds (e.g., due to an incoherent superposition of gravitational waves from a large number of unresolved white dwarf binary star systems in our own galaxy, or a “pop-corn” stochastic signal produced by

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1 $h_{100}$ almost certainly lies within the range $1/2 < h_{100} < 1$.

2 $S_{\text{gw}}(f)$ is defined by $\frac{1}{T} \int_0^T |h(t)|^2 \, dt = \int_0^\infty S_{\text{gw}}(f) \, df$, where $h(t)$ is the gravitational wave strain in a single detector due to the stochastic background signal.
gravitational waves from supernova explosions \cite{7,8,9} will require different data analysis techniques than the one we present here. (See, e.g. \cite{10,11} for a detailed discussion of these different techniques.)

In addition, we will assume that the intrinsic detector noise is: (i) stationary, (ii) Gaussian, (iii) uncorrelated between different detectors and with the stochastic gravitational wave signal, and (iv) much greater in power than the stochastic gravitational wave background.

### 1.3 Cross-correlation statistic

The standard method of detecting a stochastic gravitational wave signal is to cross-correlate the output of two gravitational wave detectors \cite{1,2,3,4,5,6}:

\[
Y_Q = \int_0^T dt_1 \int_0^T dt_2 h_1(t_1) Q(t_1 - t_2) h_2(t_2) = \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \delta_T(f - f') \hat{Q}(f) \hat{h}_2(f') ,
\]

where \(T\) is the observation time and \(\delta_T(f - f')\) is a finite-time approximation to the Dirac delta function \(\delta(f - f')\). Assuming that the detector noise is uncorrelated between the detectors, it follows that the expected value of \(Y_Q\) depends only on the cross-correlated stochastic signal:

\[
\mu = \frac{T}{2} \int_{-\infty}^{\infty} df \gamma(|f|) S_{gw}(|f|) \hat{Q}(f) ,
\]

while the variance of \(Y_Q\) is dominated by the noise in the individual detectors:

\[
\sigma^2 \approx \frac{T}{4} \int_{-\infty}^{\infty} df P_1(|f|) |\hat{Q}(f)|^2 P_2(|f|) .
\]

\((P_1(|f|)\) and \(P_2(|f|)\) are again one-sided power spectral densities.) The integrand of Eq. (7) contains a factor \(\gamma(f)\), called the overlap reduction function \cite{3}, which characterizes the reduction in sensitivity to detecting a stochastic background due to: (i) the separation time delay, and (ii) the relative orientation of the two detectors. (For coincident and coaligned detectors, \(\gamma(f) = 1\) for all frequencies.) Plots of the overlap reduction function for correlations between LIGO Livingston and the other major interferometers and ALLEGRO are shown in Fig. 1.

### 1.4 Optimal filter

Given Eqs. (7) and (8), it is relatively straightforward to show that the SNR \((\mu/\sigma)\) is maximized when

\[
\hat{Q}(f) = \frac{\lambda \gamma(|f|) S_{gw}(|f|)}{P_1(|f|) P_2(|f|)} \propto \frac{\gamma(|f|) \Omega_{gw}(|f|)}{|f|^2 P_1(|f|) P_2(|f|)} ,
\]

where \(\lambda\) is a (real) overall normalization constant. Such a \(\hat{Q}(f)\) is called the optimal filter for the cross-correlation statistic. For such a \(\hat{Q}(f)\), the expected SNR is

\[
\text{SNR} \approx \frac{3H_0^2}{10\pi^2} \sqrt{T} \left[ \int_{-\infty}^{\infty} df \frac{\gamma^2(|f|) \Omega_{gw}^2(|f|)}{|f|^6 P_1(|f|) P_2(|f|)} \right]^{1/2} ,
\]

which grows like the square-root of the observation time \(T\).

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\(3\) \(\delta_T(f) := \int_{-T/2}^{T/2} dt e^{-i2\pi ft} = \sin(\pi f T)/\pi f\).
Figure 1: Overlap reduction function between LIGO Livingston and the other major interferometers plus ALLEGRO (in an optimal alignment of 72° East of North).
1.5 Time-shifted data

If the time-series data $h_1(t)$ and $h_2(t)$ are shifted in time relative to one another, the cross-correlation statistic $Y_Q$ will depend on this shift according to:

$$Y_Q(\tau) = \int_0^T dt_1 \int_0^T dt_2 h_1(t_1 + \tau) Q(t_1 - t_2) h_2(t_2). \quad (11)$$

Making a change of variables $\bar{t}_1 = t_1 + \tau$, we have

$$Y_Q(\tau) = \int_0^T d\bar{t}_1 \int_0^T dt_2 h_1(\bar{t}_1) Q[(\bar{t}_1 - t_2) - \tau] h_2(t_2) \quad (12)$$

$$= \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} d'f' \delta_T(f - f') \tilde{h}_1^*(f) \tilde{Q}(f') e^{i2\pi f'\tau} \tilde{h}_2(f'), \quad (13)$$

which is the same as Eq. (6) with $\tilde{Q}(f)$ replaced by $\tilde{Q}(f) e^{i2\pi f\tau}$. The expected value is thus (c.f. Eq. (7))

$$\mu(\tau) = \frac{T}{2} \int_{-\infty}^{\infty} df \left| f \right| S_{gw}(\left| f \right|) \tilde{Q}(f) e^{i2\pi f\tau}, \quad (14)$$

which is simply the inverse Fourier transform of

$$\tilde{\mu}(f) := \frac{T}{2} \left| f \right| S_{gw}(\left| f \right|) \tilde{Q}(f) = \frac{T}{20\pi^2} \frac{3H_0^2}{|f|^{-3}} \Omega_{gw}(\left| f \right|) \tilde{Q}(f). \quad (15)$$

This is a useful result since $\mu(\tau)$ tells us how the mean value of the cross-correlation statistic changes with time lag.

1.6 Observational constraints

(i) The strongest observational constraint on $\Omega_{gw}(f)$ comes from the high degree of isotropy observed in the CMBR. The one-year[12, 13], two-year[14], and four-year[15] data sets from the Cosmic Background Explorer (COBE) satellite place very strong restrictions on $\Omega_{gw}(f)$ at very low frequencies:

$$\Omega_{gw}(f) h_{100}^2 \leq 7 \times 10^{-11} \left( \frac{H_0}{f} \right)^2 \quad \text{for} \quad H_0 < f < 30H_0. \quad (16)$$

Since $H_0 \approx 3.24 \times 10^{-18} \, h_{100} \, \text{Hz}$, this limit applies only over a narrow band of frequencies ($10^{-18} \, \text{Hz} < f < 10^{-16} \, \text{Hz}$), which is far below any frequency band accessible to investigation by either earth-based (10 Hz $\lesssim f \lesssim 10^3$ Hz) or space-based (10$^{-4}$ Hz $\lesssim f \lesssim 10^{-1}$ Hz) detectors.

(ii) Another observational constraint comes from roughly a decade of monitoring the radio pulses arriving from a number of stable millisecond pulsars[16]. These pulsars are remarkably stable clocks, and the regularity of their pulses places tight constraints on $\Omega_{gw}(f)$ at frequencies on the order of the inverse of the observation time of the pulsars ($\sim 10^{-8} \, \text{Hz}$):

$$\Omega_{gw}(f = 10^{-8} \, \text{Hz}) h_{100}^2 \leq 10^{-8}. \quad (17)$$

Like the constraint on the stochastic gravitational wave background from the isotropy of the CMBR, the millisecond pulsar timing constraint is irrelevant for earth-based and space-based detectors.
(iii) The third and final observational constraint on $\Omega_{gw}(f)$ comes from the standard model of big-bang nucleosynthesis\cite{17}. This model provides remarkably accurate fits to the observed abundances of the light elements in the universe, tightly constraining a number of key cosmological parameters. One of the parameters constrained in this way is the expansion rate of the universe at the time of nucleosynthesis. This places a constraint on the energy density of the universe at that time, which in turn constrains the energy density in a cosmological background of gravitational radiation:

$$\int_{f>10^{-8} \text{ Hz}} \ln f \Omega_{gw}(f) h_{100}^2 \leq 10^{-5} .$$

This constraint corresponds to a 95% confidence upper bound on $\Omega_{gw}(f)$ of roughly $10^{-7}$ in the frequency band of earth-based interferometers.

1.7 Upper-limits

In addition to the above observational constraints, there are a couple of (much weaker) upper-limits on $\Omega_{gw}(f)$ that have been set directly using gravitational wave data: (i) An upper-limit from a correlation measurement between the Garching and Glasgow prototype interferometers\cite{18}:

$$\Omega_{gw}(f) h_{100}^2 \leq 3 \times 10^5 \quad \text{for} \quad 100 < f < 1000 \text{ Hz} ,$$

(ii) An upper-limit from data taken by a single resonant bar detector\cite{19}:

$$\Omega_{gw}(f = 907 \text{ Hz}) h_{100}^2 \leq 100 .$$

(iii) An upper-limit from a correlation measurement between the EXPLORER and NAUTILUS resonant bar detectors\cite{20, 21}:

$$\Omega_{gw}(f = 907 \text{ Hz}) h_{100}^2 \leq 60 .$$

Note that these last two upper-limits are for $\Omega_{gw}(f)$ evaluated at a single frequency ($f = 907$ Hz), which is near the resonant frequency of the bar detectors.
References


