Controlling unboundedness in the gravitational path integral

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New results are presented on the Euclidean path-integral formulation for the partition function and density of states pertinent to spherically symmetric black-hole systems in thermodynamic equilibrium. We extend the path-integral construction of Halliwell and Louko which has already been used by one of us (Louko and Whiting), and investigate further a lack of uniqueness in our previous formulation of the microcanonical density of states and in the canonical partition function. In that work, the method chosen for removing the ambiguity resulted in two specific path-integral contours having finite extent. Physically motivated criteria exercised a dominant influence on that choice, as did the need to overcome the unboundedness from below of the gravitational action. The new results presented here satisfy the same physical criteria, but differ in ways which are physically significant. The unboundedness is not now eliminated directly but, for positive temperatures only, it is dealt with by what may be viewed as the introduction of an effective measure, which nevertheless may be of exponential order. Having chosen to investigate alternative contours which, in fact, have infinite extent, we find that imposing the Wheeler-DeWitt equation automatically selects out particular finite end points for the contours, at which the singularity in the action is canceled. A further important outcome of this work is the emergence of a variational principle for the black hole entropy, which has already proved useful at the level of a zero-loop approximation to the coupling of a shell of quantum matter in equilibrium around a Schwarzschild black hole (Horwitz and Whiting). In the course of enquiring into the nature of the variables in which the path integral is constructed and evaluated, we were able to see how to give a unifying description of several previous results in the literature. A concise review of these separate approaches forms an integral part of our new synthesis, relating their various underlying ideas on Hamilton-Jacobi theory and Hamiltonian reduction in the context of path integration. The new insight we gain finally helps motivate the choice of the integration variables, identification of which has played an important role in our whole analysis.

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I. INTRODUCTION

One of the most powerful tools for investigating the quantum properties of any physical system, including one in which there are self-gravitational interactions, is the Feynman path integral. This is apparently the case despite the general mathematical difficulty of determining a well-defined procedure by which formally to construct any particular path integral. Rather than preoccupation with the question of resolving quite general issues, in this paper we will concentrate on specific aspects of the nonperturbative quantum treatment of a theory such as general relativity. In handling the gravitational degrees of freedom, apart from the details of choosing gauge conditions (in response to the general covariance of the theory), and in dealing with variables which have restricted physical ranges (certainly in connection with the signature of the metric), there have also arisen problems related to finding a specification of appropriate contours in the course of constructing the actual path integral. An ever growing effort is being made to investigate and overcome the ensuing subtleties, and in this paper we successfully contribute to that quest, by invoking the Wheeler-DeWitt equation to govern the selection of specific contours in the context of gravitational thermodynamics, but in a way which may have implications also in the cosmological context.

Gravitational thermodynamics continues to be an area of fruitful investigation partly because the thermodynamics is so well understood, and partly because such simple systems as (Lorentzian) static, spherically symmetric black-hole spacetimes yield extremely interesting and nontrivial thermodynamic results. Through a variety of uniqueness theorems it has been established that, classically, black-hole spacetimes are entirely characterized by a few degrees of freedom which, in a quantum physics sense, appear to behave essentially like a (relativistic) particle system in several variables. While there remain an infinity of graviton degrees of freedom associated with the propagation of gravitational waves which might disturb stationary black-hole configurations, the treatment of these is not so dissimilar from that for waves of lower spin in a curved geometry (in principle; though it is obviously more difficult and more complicated in practice, and in a perturbative sense less satisfactory because of the inherent nonrenormalizability of the theory). However, there is still some residual ambiguity arising from the

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quantum treatment of even such restricted gravitational systems as those in which these propagating degrees of freedom are absent. In this paper we find a resolution of that ambiguity which finally leads to a maximum principle for black-hole entropy, a result which has been remarkably elusive throughout previous investigations of that issue and the related question of the density of black-hole quantum states.

Much of what we now know of the physics of simple microscopic systems, after more than 60 years since the introduction of quantum mechanics, has been obtained by an interesting interplay between purely classical and purely quantum ideas, and has led to the emergence of a whole new area of semiclassical thinking and development. The notion that a sufficiently large or sufficiently energetic system should be dominated by essentially classical considerations (perhaps subject to quantum conditions) has played a recurring role in this development, although we now know of such phenomena as superconductivity, the quantum Hall effect, the Josephson junction, and heavy electron systems, for example, which are determined by genuinely macroscopic quantum effects. General relativity still represents a fairly new arena in which the behavior, even of large systems, may yet be determined by the principles of quantum physics.

Progress in gravitational thermodynamics has already borne the fruits of an intricate interplay between classical and quantum arguments and, as exemplified by the close links drawn between these alternative avenues in the following section, it continues to do so. A striking example of this early progress [1] led to the original realization that there should be an outgoing flux of radiation at finite temperature following gravitational collapse to a black hole. Subsequent semiclassical discussion has relied on there being some domain in which the equilibrium partition function for an eternal black hole is dominated by a simple, locally minimal, classical Euclidean action [2]. It has also been understood that the relevant system might nevertheless undergo a "quantum" induced phase transition, when this extremum failed to be a global minimum. Hamiltonian reduction has represented a convenient avenue by which to get beyond a purely stationary point analysis and comprehend this semiclassical result [3]. Most recently a combination of classical and quantum criteria has been used to guide the selection of crucial integration contours in a path-integral formulation of the partition function [4]. That work incorporated existing results on local and global stability, and has led to a specification for the density of states in the microcanonical ensemble, well outside the classical domain. One salient feature of this density of states is that it was cut off to zero above a certain energy, a property whose existence had been hinted at in previous work [5].

A. Overview

In this paper, we continue to consider a path-integral construction of the quantum theory for an Einstein action reduced to a finite number of degrees of freedom. However, before proceeding to the specification of the path integral in the context of black-hole thermodynamics, we review an appropriate Hamilton-Jacobi formulation and several different (Hamiltonian-)reduction schemes for this constrained dynamical system. Our synthesis brings out a number of relations between previous results and thereby leads eventually to a new maximum variational principle for gravitational entropy. Such an outcome, which essentially had been lost in obtaining the result of Ref. [4], does not depend for its existence on any specific details of the path integration. Additionally, this work partly motivates the choice of variables used previously in the path integral evaluation and provides support for the practical use of both classical and semiclassical results in gravitational physics.

Using essentially the same starting point for the formulation of the path integral as in Ref. [4] but, after some further investigation, by making a different choice for the crucial integration contours, we have been able to satisfy the previous physical criteria and yet arrive at a new specification of the density of states, which is physically distinct in that it remains nonzero to arbitrarily high energies. We show how both Hamilton-Jacobi analysis and a specific non-Hamiltonian-reduction hypothesis lead to essentially the same starting point (up to a simple measure), as is obtained by more elaborate path-integral methods. In addition, by referring back to the traditional difficulties associated with the quantization of gravitating systems, we can throw new light on control of the unboundedness (from below) problem for the gravitational action, a problem which actually arises here in a natural way and is dealt with, by effectively being cut off, in a very unusual and decisive manner. This result is obtained essentially by invoking the Wheeler-DeWitt equation for the partition function while determining the finite end point of a contour integral, which turns out to be identical to the Laplace transform of the density of states, which we subsequently identify. In this sense the Laplace transform has been accommodated directly into the path integral. From a consideration of the inverse Laplace transform of our eventual result we are able, in retrospect, to point to a fundamentally new zero-loop variational principle in which the concomitant gravitational entropy is globally maximized. Our new prescription for the density of states in the microcanonical ensemble remains nonzero at arbitrarily high energies, in contrast with that found in Ref. [4], which became cut off to zero above some finite maximum energy depending on the thermodynamic boundary data.

To direct the reader's attention to the results obtained here and to their relation with other recent work, we summarize the layout of the remainder of the paper, before proceeding. After a brief restatement of the appropriate classical variational problem in Sec. II, and then reviewing recent developments in the thermodynamic context, we go on to reconsider details of Hamilton-Jacobi solutions, especially in connection with the path-integral formulation. We examine the relation between several semiclassical treatments and identify the starting point which is obtained directly from the path-integral work of Halliwell and Louko [6]. In Sec. III, by investigating the choice of contours available we find that the unboundedness of the gravitational action can actually be controlled in a satisfactory way. Section IV contains a brief discussion of various consequences of our investigation, including properties of the new density of states which we obtain, the emergence of a zero-loop variational principle in which the gravitational entropy is maximized, and implications for (using the Wheeler-DeWitt equation to control) the generic unboundedness of the action.

II. A UNIFYING PERSPECTIVE

It has been established that black holes have both an entropy [7] and a temperature [1], both of which constitute results in quantum physics, even if formulated in a statistical or thermodynamic domain. Since their introduction into the statistical and thermodynamic study of gravitational systems [8,9], path-integral methods have been productively employed, even in investigations examining the quantum properties of bounded systems containing a black hole [3,4]. A recurrent problem plaguing efforts in this area generally, but which was effectively absent in Refs. [3,4], concerns the unboundedness from below of the gravitational action.

The work of Ref. [5] on the black-hole density of states, even though only a zero-loop result, effectively showed that this unboundedness might, after all, be controlled for finite black-hole systems, since an intelligible finite result was obtained, but it gave no hint of a method by which that control might be exercised, in general. In the following section we will find that the Wheeler-DeWitt equation may have a role to play in determining the circumstances of such control, and perhaps in a much wider context than that considered here. In the meantime, in order to eventually see how this control may come about, we must begin by obtaining a specification of the canonical partition function governing the thermodynamics of finite black-hole systems.

By following the Feynman prescription, the partition function for the canonical ensemble can be given a formal path-integral representation. In gravitational thermodynamics, such a path integral would be evaluated over Euclidean four-geometries on a manifold with the requisite topology, and would take the form

$$Z(\gamma)=\int \mathcal{D}[g]\exp(-I[g]) , \tag{1}$$

in which the induced metric γ on the boundary is held fixed, corresponding to the canonical thermal boundary conditions of fixed inverse temperature, β, at walls of fixed area A. For each problem in statistical mechanics in which there arises such geometrically canonical data there is, in fact, an associated well-posed boundary-value problem in general relativity [10]. In particular, with no matter present, *all* relevant thermodynamic data are purely geometrical.

In principle, the path integral requires a sum over all geometries obeying the prescribed boundary data. However, of technical necessity, in minisuperspace applications, this space of geometries has been reduced to the smaller domain consisting only of static, spherically symmetric geometries, or, even further, to just those satisfying the Hamiltonian constraint, *always* on manifolds with the requisite topology. In this way it has been shown that, under conditions properly defining a suitable canonical ensemble, a bounded black-hole system may exist in stable thermal equilibrium [2]. To indicate the basis of such an understanding and subsequently, for a more complete investigation of the underlying path integral, we turn now to a description of the classical Euclidean theory of general relativity.

A. Classical variational principle[1]

In line with our intention to treat only the "nonpropagating" gravitational degrees of freedom, we will concentrate here on static, spherically symmetric geometries and adopt the following form for the metric:

$$ds^2=a^2(y)d\tau^2+N^2(y)dy^2+b^2(y)d\Omega^2 , \tag{2}$$

defined on manifolds having topology $M=D\times S^2$, and thereby the Euler characteristic, $\chi=2$. The disk D is mapped by the polar coordinates (y,τ) with $0<y\leq 1$ and τ has a period of 2π. The metric of a unit two-sphere is denoted by $d\Omega^2$. We shall generally adopt units in which G and $\hbar$ are 1. The manifolds we consider have a boundary at $y=1$ with topology $\partial M=S^1\times S^2$. The intrinsic metric of the boundary is characterized by $a(1)$ and $b(1)$, which have a thermodynamical interpretation as the inverse temperature of the wall, $\beta=2\pi a(1)$, and its area, $A=4\pi b^2(1)$. We are interested in constructing the partition function for the canonical ensemble in which the pair of data (β, A) are held fixed. The appropriate classical action is therefore the one which is stationary when the field equations hold and the boundary metric is fixed. Such an action is the modified Hilbert-Einstein action [8,11]

$$I=-\frac{1}{16\pi}\int_M R\sqrt{g}\,d^4x+\frac{1}{8\pi}\int_{\partial M}(K-K^0)\sqrt{\gamma}\,d^3x , \tag{3}$$

where K is the trace of the extrinsic curvature of the boundary and the constant K^0 is chosen to be the trace of the extrinsic curvature of the boundary embedded in flat Euclidean space. This choice is of some consequence even for the classical theory, in which it determines the reference level of energy, and will also affect the quantum theory, though not the entropy to any order. For metrics of the form (2) the action may be expressed as

$$I=-\pi\int_0^1\left[\frac{ab'^2+2ba'b'}{N}+aN\right]dy \\ +2\pi a(1)b(1)-\pi b^2(0) , \tag{4}$$

where a prime denotes $\partial/\partial y$, and $y=0$ physically locates the black-hole horizon. In the course of extremizing the action given by (4) we will be forced to deal with conditions at the coordinate singularity at $y=0$. Accordingly, we consider as part of our variational principle not only

[1]Treatment in this section parallels that in Ref. [4], and sources cited therein.

$$\delta a(1) = \delta b(1) = 0 , \tag{5}$$

but also [4]

$$a(0) = 0 , \tag{6}$$

even though $y = 0$ is not a boundary, but rather the center of the manifold M. Upon variation of (4) while respecting (5) and (6), we obtain

$$\delta I = \pi \int_0^1 \left[\frac{\delta I}{\delta a} \delta a + \frac{\delta I}{\delta N} \delta N + \frac{\delta I}{\delta b} \delta b \right] dy + 2\pi \left[\left[\frac{a'}{N} - 1 \right] b \delta b \right]_{y=0} . \tag{7}$$

When the equations

$$\frac{\delta I}{\delta a} = \frac{N}{b'} \left[b \left[\frac{b'^2}{N^2} - 1 \right] \right]' = N b^2 G_0^0 = 0 , \tag{8}$$

$$\frac{\delta I}{\delta N} = \left[\frac{a b'^2 + 2 b a' b'}{N^2} - a \right] = a b^2 G_1^1 = 0 , \tag{9}$$

$$\frac{\delta I}{\delta b} = 2 \left[\left[\frac{(ab)'}{N} \right]' - \frac{a' b'}{N} \right] = a N G_2^2 = a N G_3^3 = 0 , \tag{10}$$

$$\left[\frac{a'}{N} - 1 \right]_{y=0} = 0 \tag{11}$$

are satisfied, $\delta I = 0$. Equation (11) is the condition of regularity at the center, so that solutions to our classical field equations are Euclidean Schwarzschild black holes, with merely a coordinate singularity at $y = 0$ in our chosen form of the metric.

B. Hamiltonian reduction[2]

One of the characteristic properties of general relativity is that, through it, the energy in a spacelike hypersurface can be evaluated by a surface integral. Consequently, in a static geometry such as we are considering, when the Hamiltonian constraint is satisfied, up to topological contributions the action is given entirely by a boundary term. The dynamical Hamiltonian H_D is proportional to the G_0^0 component of the Einstein tensor (8). For smooth geometries, as dealt with in Ref. [3], its solution is given by

$$\frac{b'(y)}{N(y)} = \left[1 - \frac{r_+}{b(y)} \right]^{1/2} , \tag{12}$$

where r_+, which physically determines the black-hole horizon area (by $A = 4\pi r_+^2$), will be used throughout in place of $b(0)$. The reduced action, obtained by substituting this solution of the Hamiltonian constraint into (4), is given by

$$I^* = I|_{G_0^0=0} = \beta r_0 \left[1 - \left[1 - \frac{r_+}{r_0} \right]^{1/2} \right] - \pi r_+^2 , \tag{13}$$

in which $r_0 \equiv b(1)$. This action is exact for smooth, static, spherically symmetric Euclidean geometries which, however, are generally not classical solutions to the full vacuum Einstein equations, except possibly at stationary points with respect to r_+ variations.

For the purposes of defining a path integral, we can imagine the path measure to be chosen so that the constraint (8) is enforced through δ-function conditions at each point. With reference to the smooth geometries presently under discussion, the path integral should then become simply an ordinary integral in r_+, for example [3],

$$Z^*(\beta, A) = \int_0^{r_0} \mu(r_+) \exp[-I^*(\beta, r_0; r_+)] dr_+ , \tag{14}$$

but evaluation of the residual "effective" measure $\mu(r_+)$, arising from very irregular geometries in the neighborhood of each smooth geometry for which we have evaluated the action in (13), must now be obtained by an independent argument. Such an argument was given in Ref. [3], which allowed a specification of the partition function to be completed. Even without this result, the Wheeler-DeWitt equation was satisfied to leading order. Moreover, whenever classical stationary points dominate the path integral, the properties of the resulting partition function depend principally on an analysis of $I^*(\beta, r_0; r_+)$ and not on details of the adopted measure. In particular, for $\beta/r_0 < 32\pi/27$, the global minimum of the action with respect to variations in r_+ corresponds to a Schwarzschild black-hole solution (as well as to an infinity of nonvacuum geometries with the same value of the action), and it uniquely determines the zero-loop free energy of a thermodynamically stable equilibrium state in the canonical ensemble with the given boundary data. For larger β/r_0, the path integral is dominated by nonclassical geometries, despite the existence of classical stationary points whenever $\beta/r_0 < 8\pi/\sqrt{27}$. In fact, two real Euclidean solutions to the full vacuum equations occur for *all* data in this range [2].

C. Hamilton-Jacobi solutions[3]

Hamilton-Jacobi analysis played an important part in the original development of a quantum-mechanical understanding of microscopic systems. Analogously, for macroscopic black-hole systems, we have found that there is a subtle and rather interesting relationship between the reduced action (13) and a solution to the Hamilton-Jacobi equation associated with the classical action (4). It will be seen from the form of the action (4), that the Euclidean geometries corresponding to static, spherically symmetric systems in thermal equilibrium most naturally foliate in a radial direction. It is with respect to this foliation that we are interested in applying the Hamilton-Jacobi method. Because its solution will also bear on the form in which we eventually consider the path integral, and on the action arising from an alterna-

[2]A more complete analysis can be found in Ref. [3].

[3]This same problem arose in a physically unrelated cosmological context; see Ref. [12].

tive method of reduction which we investigate in the next section, we will briefly summarize the relevant Hamilton-Jacobi results.

To simplify the analysis we rewrite the implied Lagrangian in (4) by replacing the variable a with a new variable, $q=a^2b$, and we consider just that part of the action which is without the additive constants:

$$W=-\pi\int_0^1\left[\left[\frac{b}{q}\right]^{1/2}\frac{q'b'}{N}+\left[\frac{q}{b}\right]^{1/2}N\right]dy$$
$$=\int_0^1(P_q q'+P_b b'-Nh)dy\ . \tag{15}$$

The second line follows from defining momenta in the usual way, $P_x=\partial L/\partial x'$; and the natural Hamiltonian associated with this problem, in which the parameter y plays the role of "time," is given by $H_N=Nh$, where

$$h=-\pi\left[\frac{q}{b}\right]^{1/2}\left[\frac{P_q P_b}{\pi^2}-1\right]. \tag{16}$$

There is also a constraint $P_N=0$. The Hamilton-Jacobi equation is obtained by replacing P_x by $\partial W/\partial x$, and setting

$$H_N\left[q,b;\frac{\partial W}{\partial q},\frac{\partial W}{\partial b}\right]=0\ . \tag{17}$$

Solutions which relate to the reduced action of the previous section can be obtained by the method of separation of variables in the form $\overline{W}=Q(q)B(b)+C$. Such a solution is

$$\overline{W}=2\pi[BQ-\sqrt{b(1)-b(0)+B^2/c}\sqrt{q(1)-q(0)+cQ^2}], \tag{18}$$

where B and Q are constants of integration and c is the separation constant. (c could, of course, be scaled away; but that is of no real consequence here.) For geometries satisfying (6) with our chosen boundary data, $\overline{W}$ is maximal in c, for example, with a value which leads directly to the reduced action (13).

It is also clear that solutions may readily be obtained by the method of separation of variables in the alternative form $W=Q(q)+B(b)$, which is much closer to the form traditionally used in solving classical problems, and for establishing a relationship between classical and quantum physics. A solution in this form is given by

$$W=\frac{-\pi}{\alpha}[q(1)-q(0)]-\pi\alpha[b(1)-b(0)]\ , \tag{19}$$

where α is the separation constant. (An identical solution occurs when the lapse rescaling of Ref. [6] is adopted.) Imposing (6) and reintroducing the additive constants leads directly to the reduced action we obtain in the next section. We shall also see that, subject to an appropriate definition of the lapse, this is exactly the action which arose in the path integral of Ref. [4]. Finally, we note that the action obtained by reduction using the dynamical Hamiltonian in the previous section again results, as an envelope solution, when (19) is evaluated at its stationary point with respect to α.

D. Alternative reduction

Whereas the constraint imposed in Ref. [3] corresponded to the dynamical Hamiltonian H_D of a timelike foliation, it seems reasonable to investigate the consequences if, instead, the domain of the path integral were to be reduced by imposing the Hamiltonian constraint associated with the natural foliation, i.e., the H_N which generated the Hamilton-Jacobi solutions, and which is proportional to G_1^1. The equation $G_1^1=0$ cannot be solved directly, although it could be substituted back into the action before proceeding. Reduction by this method will be considered elsewhere [13].

Here we will consider, as an alternative approach, reduction by the "constraint," $G_0^0-G_1^1=0$, which is known to be solvable; in fact, it frequently is used to set $g^{00}g^{11}=1$ for static spherically symmetric geometries. Perhaps the strongest motivation for using this alternative condition comes after the fact, from the emergence of results which relate to our Hamilton-Jacobi discussion. By an appropriate identification of variables, and up to a simple measure term, this reduced action was encountered in the path integral of Ref. [4], once the canonical path integral had been reduced to an ordinary double integral over two remaining parameters. Geometrically one of these parameters still corresponds to the location of the event horizon, r_+, while the other, already denoted by α, apparently corresponds to the inverse Hawking temperature (via the surface gravity), and arose as the separation constant in the Hamilton-Jacobi problem.

In terms of the new variable, $\alpha(y)=Na/b'$, and the variable q, we may recast (4) into the form

$$I=-\pi\int_0^1\left[\frac{(q)'}{\alpha}+\alpha b'\right]dy+2\pi a(1)b(1)-\pi b(0)^2\ . \tag{20}$$

Furthermore, from (8) and (9), we see that the equation, $G_0^0-G_1^1=0$, may be written as

$$G_0^0-G_1^1=2\frac{a}{Nb}\left[\frac{1}{\alpha}\right]'=0\ , \tag{21}$$

which has the solution $\alpha=\text{const}$, a constant equivalent to the separation constant in the Hamilton-Jacobi solution. When condition (21) is satisfied, we immediately obtain

$$I|_{G_0^0-G_1^1=0}=-\frac{\pi}{\alpha}[q(1)-q(0)]-\pi\alpha[b(1)-b(0)]$$
$$+2\pi a(1)b(1)-\pi b^2(0)$$
$$=-\frac{1}{\alpha}\frac{\beta^2 r_0}{4\pi}-\pi\alpha(r_0-r_+)+\beta r_0-\pi r_+^2\ , \tag{22}$$

where use of (6) has been made in the last step. In this case, the path integral is reduced to a double integral

$$Z(\beta,A)=\int\tilde{\mu}(r_+,\alpha)\exp[-I(\beta,A,r_+,\alpha)]dr_+d\alpha\ , \tag{23}$$

with some appropriate measure $\tilde{\mu}(r_+,\alpha)$. Precisely this result, for

$$\tilde{\mu}(r_+,\alpha)=\mu(r_+)/\alpha \ , \tag{24}$$

was obtained in Ref. [4], and we now turn to a consideration of that work.

E. Characterizing the partition function[4]

The partition function of Ref. [4] was obtained from a path integral evaluated according to a prescription of Ref. [6]. In the course of constructing that path integral it had been found necessary to formulate it in terms of a rescaled lapse given by

$$\tilde{N}=aN \ . \tag{25}$$

With a path measure *defined* in terms of b, q, and $\tilde{N}$ at each point, the bulk of the path integral could then be evaluated by Gaussian integration in the variables b and q. There remained the lapse integration which, in the "proper-time" gauge, reduced to a single ordinary integral in $\tilde{N}$, but the contour for this integral had not been predetermined merely by the formulation. Results for several choices of the lapse contour were tabulated in Ref. [6], but none of them included integration along the positive real axis because the integral was divergent at both ends of that contour, a consequence of the gravitational action being unbounded from below.

It will be convenient to turn our remaining discussion of this analysis into a form suitable for reference in Sec. III. Thus, before performing the final $\tilde{N}$ integration, we combine the intermediate result of Ref. [6] with the additional integral over $b(0)(\equiv r_+)$ required to complete the specification of a partition function. This leads us to consider the "action" which can be expressed in our variables as

$$I=\frac{-\pi}{\tilde{N}}[q(1)-q(0)][b(1)-b(0)] \\ -\pi\tilde{N}+2\pi a(1)b(1)-\pi b^2(0) \ . \tag{26}$$

We note that there actually exist four-geometries, depending on the boundary data $b(1)$ and $q(1)$ and on the additional parameters $\tilde{N}$ and $b(0)$, for which the action is exactly that given in (26). Because the resultant lapse measure, $d\tilde{N}/\tilde{N}$, appearing in Ref. [6] is scale invariant, integration in the lapse can immediately be related to an integral involving the reduced action of the previous section with a lapse reparameterized in terms of the variable α by

$$\alpha=\frac{q(1)-q(0)}{\tilde{N}} \tag{27}$$

(or, equally, by $\alpha=\tilde{N}/[b(1)-b(0)]$; cf. previous section). Given the table of results in Ref. [6], the task faced in Ref. [4] was to select contours corresponding to the α and r_+ integrals so that a quantity might be obtained having the properties appropriate to a partition function for the canonical ensemble presently under discussion, for which the relevant stationary points should always be on the positive real axis. These properties include the requirement that, when it is dominated by a stationary point of the action, that point should be at least a local minimum. In addition, there should be no other classical solution with the given boundary data for which the value of the action is less. Every partition function should also have the form of a Laplace transform related to the variable β, and in the present situation must satisfy the Wheeler-DeWitt equation, as any path integral should do.

There is a local minimum in the reduced action (13) only for $\beta/r_0<8\pi/\sqrt{27}$, and at this local minimum r_+ satisfies: $2r_0/3<r_+<r_0$. The determinant of the Hessian of I at the stationary point with respect to α and r_+ variations is negative for data leading to r_+ in this range, indicating that one of the α and r_+ contours must be purely imaginary at the physically relevant stationary point. Only one lapse contour in the table of Ref. [6] passed through the positive stationary point, and it was a closed contour around the origin which crossed the real axis in the imaginary direction. Thus the r_+ integral was required to be real, but it would diverge unless cut off at a finite point. That point could depend on the data, but only in a certain way if the partition function was to satisfy the Wheeler-DeWitt equation.

In the variables we have adopted, the Wheeler-DeWitt equation takes the form

$$\frac{1}{\pi}\left[\frac{q}{b}\right]^{1/2}\left\{-\left[\frac{\partial}{\partial q}+\pi\left[\frac{b}{q}\right]^{1/2}\right]\right. \\ \left.\times\left[\frac{\partial}{\partial b}+\pi\left[\frac{q}{b}\right]^{1/2}\right]+\pi^2\right\}Z=0 \ , \tag{28}$$

where account has been taken of the shift in the canonical momenta caused by the K^0 term in the action. It can be directly verified that the limits zero and r_0, chosen for the r_+ integration in Refs. [3,4], allow the partition function, based on the reduced action (26) modified by (27), to satisfy the Wheeler-DeWitt equation exactly, provided the "effective" measure μ can be found independent of the data (i.e., β,r_0), as indicated in (24) and analogous to (14). A key feature of the density of states deduced from this formulation of the partition function is that it was nonzero only for E in the range $0<E<2r_0$. The partition function itself was found to be dominated by a classical solution *only* when it arises for a minimum of the action which is negative, consistent with the result previously obtained in Refs. [2,3]. Somewhat incomprehensibly, the finite support of the ensuing density of states meant that quantum states of the gravitational field could be defined with arbitrarily negative temperature, a curiosity which had already emerged in the discussion of Ref. [5], but still with no prescription for how to prepare such states.

III. INVESTIGATING THE PATH INTEGRAL

Using the path-integral "action" (26) obtained in the previous section, and expressing it in terms of our new variables and our chosen boundary data, we find that the definition of the partition function adopted in Ref. [4] can be rewritten as

[4]Our discussion rests heavily on that in Ref. [4], using results from Ref. [6].

$$Z(\beta,r_0)=\frac{1}{2\pi i}\int\frac{d\alpha}{\alpha}\int\mu(r_+)dr_+\exp\left[-\beta r_0+\frac{\beta^2 r_0}{4\pi\alpha}+\pi\alpha(r_0-r_+)+\pi r_+^2\right], \tag{29}$$

in which, previously [4], the α integral was carried out along a closed loop about the origin (resulting in a zeroth order, modified Bessel function) and r_+ was integrated from zero to r_0. With these prescribed contours, the integrals can be evaluated in either order. For concreteness, in this section, we will make the specific choice $\mu(r_+)=2\pi$ in Planck units, before proceeding.

We have examined alternative choices for the contours guided, as previously, by the requirement that the partition function be a Laplace transform which satisfies the appropriate Wheeler-DeWitt equation and that, in the classical regime, it should be dominated by a thermodynamically stable black-hole solution. In particular, we have investigated using the entire imaginary axis for the r_+ integration, and the positive real axis for the α integration, a choice which preserves the relative properties of the contours which were identified as essential in the previous section. In general, some modification of this choice is necessary in order to deal with the singularity in the exponential at $\alpha=0$. From among a number of options considered, we believe that shifting the r_+ contour to the right and cutting it off from below at a particular point represents the most acceptable resolution available. The cutoff point is chosen so that the Wheeler-DeWitt equation will be satisfied while nevertheless ensuring that the required classical solution should dominate, at least for positive β. With the choice we make, it has also been possible to determine that, just for β positive, there will be no residual singularity in the exponential at $\alpha=0$.

The appropriate point at which to cut off the r_+ integral is found to be

$$r_c=\frac{\alpha}{2}-i\sqrt{\alpha r_0}\left[1-\frac{\beta}{2\pi\alpha}\right]. \tag{30}$$

Our result is then [14]

$$\begin{aligned}Z(\beta,r_0)&=\frac{1}{i}\int_0^\infty\frac{d\alpha}{\alpha}\int_{r_c}^{\alpha/2+i\infty}dr_+\exp\left[-\beta r_0+\frac{\beta^2 r_0}{4\pi\alpha}+\pi\alpha(r_0-r_+)+\pi r_+^2\right]\\&=\frac{1}{2}\int_0^\infty\frac{d\alpha}{\alpha}\operatorname{erfc}\left[-\sqrt{\pi\alpha r_0}\left[1-\frac{\beta}{2\pi\alpha}\right]\right]\exp\left[-\beta r_0+\frac{\beta^2 r_0}{4\pi\alpha}+\pi\alpha r_0-\frac{\pi\alpha^2}{4}\right].\end{aligned} \tag{31}$$

It may appear that the α integral must be evaluated last in this expression for $Z(\beta,r_0)$, but such is not the case, as is shown by the subsequent expression in (33) below, which may be obtained from (31) by a change of integration variable.

From Ref. [14] we can see that, as $\alpha\to\infty$, erfc$\to 2$ and the integral in (31) is clearly well behaved there. This same asymptotic limit applies for large classical systems, which occupy stationary points of the exponent. Furthermore, as $\alpha\to 0$ for any finite positive β,

$$\operatorname{erfc}\to\frac{2}{\beta}\left[\frac{\alpha}{r_0}\right]^{1/2}\exp\left[+\beta r_0-\frac{\beta^2 r_0}{4\pi\alpha}-\pi\alpha r_0\right], \tag{32}$$

which cancels the singularity in the exponential and also causes the singular measure to become integrable. This latter asymptotic limit would diverge at $\beta=0$, consistent with the fact that (31) is not well defined at that point. It is clear that this Z does not exist for $\beta<0$, a data range which is available to the partition function of Ref. [4]. Thus, our new choice for the integration contours does not completely remove the singularity in the action, which can still be seen in (31), and (33) below, for negative β, but does so only for β positive. Hence, the Wheeler-DeWitt equation has helped us to separate the physically reasonable domain ($\beta>0$) from the problematical domain ($\beta<0$), for which it has remained difficult to establish an entirely satisfactory physical interpretation, or even a prescription for the preparation of an appropriate state of the gravitational field, corresponding to boundary data from this domain.

A. Density of states

Rather than attempting to derive our proposed density of states from the partition function obtained in the previous section, we instead give an alternative expression for this partition function, from which the density of states can be read off directly. Thus, by performing the E integration first (which is valid for all $\beta>0$) in[5]

$$Z(\beta,r_0)=\int_0^\infty dE\int_0^\infty\frac{d\alpha}{\sqrt{\alpha r_0}}\exp\left[2\pi\alpha E-\pi\alpha\frac{E^2}{r_0}-\frac{\pi\alpha^2}{4}-\beta E\right], \tag{33}$$

[5] Integral (33) comes from the first line of (31) by the substitution $r_+=r_c+iE\sqrt{\alpha/r_0}$.

we obtain the result (31), indicating immediately that we can write (for $E>0$)

$$\nu(E,r_0)=\int_0^\infty \frac{d\alpha}{\sqrt{\alpha r_0}}\exp\left[2\pi\alpha E-\pi\alpha\frac{E^2}{r_0}-\frac{\pi\alpha^2}{4}\right]$$
$$=\frac{(2\pi)^{1/4}}{\sqrt{r_0}}D_{-1/2}(-\sqrt{8\pi}M)\exp(2\pi M^2) \tag{34}$$

for the density of states, where $M=E(1-E/2r_0)$, and D is a parabolic cylinder function [15]. This result for ν is elementary to confirm, because of the way β enters into (33), which allows us to do the inverse Laplace transform trivially to obtain the first line of (34).

For M large and positive (i.e., for $0<E<2r_0$) we have the asymptotic result

$$\nu(E,r_0)\sim\left[\frac{G}{Mr_0}\right]^{1/2}\exp\left[\frac{4\pi GM^2}{\hbar}\right], \qquad 0<E<2r_0/G \tag{35}$$

in which the physical constants have been inserted to show the dependence on G and $\hbar$, in particular, the disappearance of ν as $G\to 0$. This result is determined, almost exclusively, by contributions in α at the scale

$$\alpha_0=4GM=4GE(1-GE/2r_0)\le 2r_0, \qquad 0<E<2r_0/G, \tag{36}$$

as is appropriate with classically dominant solutions. For fixed r_0, ν attains its maximum at $E=r_0$ (where $M=r_0/2$), beyond which it decreases, eventually to zero as $E\to\infty$.

For M large and negative (i.e., for $E\gg 2r_0$) we have alternatively

$$\nu(E,r_0)\sim\left[\frac{G}{-2Mr_0}\right]^{1/2}\sim\frac{1}{E(1-2r_0/GE)^{1/2}}, \qquad E\gg 2r_0/G, \tag{37}$$

which (amazingly) is independent of physical constants to leading order, and its form confirms why the partition function does not exist for $\beta\le 0$. Integration by parts shows that the density of states at very high energy is effectively dominated by contributions in the integral characterized by (but not precisely limited to) the scale

$$\alpha_0=\frac{\hbar}{4\pi|M|}=\frac{\hbar r_0}{2\pi GE^2(1-2r_0/GE)}\ll\frac{\hbar}{4\pi E}\ll\frac{\hbar G}{8\pi r_0}, \qquad E\gg 2r_0/G, \tag{38}$$

which would be generally microscopic, even on the Planck scale of length, for any physically reasonable system. It is interesting to note the different dependence on physical constants in the length scales, (36) and (38), which characterize the behavior as r_0 is varied from larger to smaller values at fixed E.

These asymptotic results hold within 1%, for $M\ge 1.22m_P$, i.e., above the Planck scale, while at $M=0$ exactly we have

$$\nu(0,r_0)=c/\sqrt{r_0}, \tag{39}$$

where $c\sim 1.9256$ in Planck units. Although the expression in (34) is well defined for $E<0$, this domain does not contribute to the density of states, as taking the inverse Laplace transform of (33) makes clear.

B. Controlling unboundedness in the gravitational action

The problem of the unboundedness (from below) of the Euclidean action for general relativity has long been recognized as a genuine, serious problem. Fortuitously, it does not appear for perturbations away from flat space *provided* the Hamiltonian constraints are imposed on the spacelike surfaces of a chosen foliation; and equally, it did not arise in Ref. [3], essentially for the same reason. But cosmological examples exist for which imposing the constraints will not remove this problem. In this present work the unboundedness problem is explicitly present, in general, but our choice of contours manages to control it, for positive β only, in a very effective manner. In essence, the r_+ integral can be viewed as modifying the "measure" for the α integral, and in a exponential way near the singularity, as is shown in (32). Equally, with its asymptotic behavior at large negative argument removed from the parabolic cylinder function in (34), the partition function written as a Laplace transform can be viewed as containing an effective measure which is constant in the classical regime, but is exponentially suppressed for large energies. It is interesting to note that modified measures of this kind were deliberately excluded from consideration in the path integral of reference [4].

Our result does not introduce any additional cutoff parameter, in the usual sense. Also, it has been obtained without requiring any specific renormalization, and it satisfies the Wheeler-DeWitt equation exactly. As such, it provides almost a model solution to a perennial problem. In particular, it does not simply circumvent the issue, but puts it to good physical use—in terms of identifying the range of applicable boundary data. This close interplay between equation, boundary data, and solution is a common occurrence throughout both classical and quantum physics, and in the gravitational context is almost certainly at the heart of a permanent control of the unboundedness of the action from below. In this sense, our work indicates a new direction to pursue in order to ensure that the gravitational action may remain well behaved, even in fully quantized systems.

C. Variational principle for gravitational entropy

A global maximal principle for the zero-loop entropy is really already contained in the Hamilton-Jacobi analysis of the previous section, but it can be read off most directly from the density of states in (34):

$$S(E,r_0;\alpha)=2\pi\alpha E-\pi\alpha\frac{E^2}{r_0}-\frac{\pi\alpha^2}{4} . \tag{40}$$

It is a result of some practical importance since, by itself, it already suggests that the gravitational microcanonical ensemble is well defined for all classically admissible data [4] (i.e., E and r_0 such that $M>0$), a result which is indeed borne out by (34) for the actual density of states. An application of this zero-loop result to shells of spherically symmetric vacuum is currently under investigation [16].

For data in the classical range this entropy is extremized at $\alpha=4M$, and becomes the usual black-hole entropy, $S=4\pi M^2(E,r_0)$, in which M is identified as the mass of the system at infinity. For data outside this range (i.e., for $E\gg 2r_0$), this entropy is maximal (but *not* extremal) at $\alpha=0$, reflecting the fact that the dominant geometries are of a quantum nature. The implication is that, at high energies, states exist for which collapse does not occur despite the fact that energy is confined to a region smaller than the classical horizon. One could be reminded here of the situation for the hydrogen atom, which is classically unstable to collapse; but quantum mechanically it is stable. For the relevant thermodynamic data, M is actually negative, and S attains a global maximum at negative α, suggestively indicative of a naked singularity. Fortunately, this maximum does not contribute to the density of states, which it would cause to diverge as E approaches infinity. Instead ν actually dies smoothly, according to the inverse power-law behavior given in (37).

IV. DISCUSSION

We began in this paper to formulate a unified perspective on recent work in the field of gravitational thermodynamics as it applies to bounded spherically symmetric systems containing a black hole. Our synthesis, which represents an important part of the present work, is presented in Sec. II. Recurrent throughout this approach has been the reduction of the path integral for the appropriate minisuperspace to one ordinary double (or single) integral; by restricting to geometries satisfying the dynamical or natural Hamiltonian constraint equations, by examining solutions of the associated Hamilton-Jacobi equation, and finally by detailed computation of an explicit path integral—though with some remaining contour freedom. Semiclassical methods have been shown to provide a powerful, if slightly incomplete, way of analyzing problems of this kind, and have previously been used implicitly, both in the selection of a functional measure [3], and in settling the choice of contours left over from the path-integral formulation [4]. Remarkably, we can obtain a new maximum principle for black-hole entropy. No variational principle has previously been available for the black-hole entropy, and its existence now has already proved useful in independent work on the stability of a spherical shell of matter surrounding a black hole in a state of thermal equilibrium [16].

With the reduced path integral in hand, we were able in Sec. III to obtain some important new results. In order to proceed there, our first undertaking required us to reexamine the discussion of the available choice for the remaining contours. We eventually chose contours which gave a partition function having all the desirable properties expounded in Ref. [4] but which is nevertheless different from the result found there. Our partition function satisfies the Wheeler-DeWitt equation and obeys essentially the same global stability criterion as previously demanded. And, for positive β only, it also solves the problem of including contributions from geometries for which the action is unbounded negative. These contributions represent a well established problem in general relativity which we have managed to control, obtaining a finite result for all $\beta>0$, by integrating on complex contours with a carefully defined cutoff. The nature of this control may prove significant for other work in quantum gravity, since its formulation does not depend directly on the thermodynamical context in which it is here formulated.

The density of states we obtain has the notable property that it is nonzero for all positive values of energy, whereas the density of states of Ref. [4] is zero outside the restricted range of energy, $0<E<2r_0$. Having the density of states be nonzero for large values of the energy suggests the existence of quantum states of the gravitational field which contain more energy than is required for classical collapse. They would certainly not remain stationary if allowed to evolve under the classical Einstein equations but, however apparently unlikely, these geometries nevertheless contribute to the black-hole density of states. The $1/E$ falloff rate of our result for these quantum configurations prevents the existence of black-hole equilibrium states at any negative temperature, in sharp contrast to the consequences of the compact support in energy found for the density of states in Ref. [4].

In the present work we have invoked the Wheeler-DeWitt equation to resolve ambiguities in a path-integral construction for gravitational thermodynamics. The work in Refs. [3,4], together with this work, represent three different approaches to a single problem, which requires dealing with the gravitational action and its intrinsic unboundedness from below. Our present approach could be shown to be more head-on than previously, in the sense that it does not simply avoid the issue at the outset. For almost any other choice of r_c, (31) would contain a divergent integral, and the particular value we obtain for r_c resulted from imposing the Wheeler-DeWitt equation directly and not from *a priori* elimination of the singular behavior. This resolution also has the consequence of being effective only for the more physically interesting domain of positive β, thereby eliminating the need to discuss the somewhat problematical question of how to prepare states of the gravitational field at negative temperature.

In relation to the broader context of quantum gravity, it is significant that what we have essentially demonstrated here is a new method for controlling the unboundedness of the gravitational action. The Wheeler-DeWitt equation has not previously been used for this purpose, and it remains to be further investigated whether, through it, we may eventually construct a generalizable method for such control. The nature of our result certainly does not preclude this. With reference to canonical and other formalisms, it is precisely because they are so formal that some problems are not easily seen until one makes the context entirely explicit; and equally, some problems may not be circumvented until the context is much more clearly and less formally defined. We view our work as a direct contribution to the general problem, especially valuable because it is very definitely an example, which is both simple and explicit.

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