Window Functions for LIGO Data Analysis **14 Jun 2002**

Bernard F. Whiting **University of Florida**

We describe a number of windowing functions and discuss their usefulness for LIGO data analysis. The principal feature underlying the construction of these windowing functions is control of their differentiability in order to control their fall-off behaviour at large frequency.

We will describe the windowing function first in the frequency domain. We start with a denomenator function defined by:

$$D_N^{\Delta}(\omega) = \prod_{n=1}^{N}\Big(1 - \Big(\frac{\omega\Delta}{(2n-1)\pi}\Big)^2\Big).$$

Then we introduce a shaping function defined by:

$$S_N^{\Delta}(\omega) = \frac{\cos(\omega\Delta/2)}{D_N^{\Delta}(\omega)}.$$

This has the property that it is unity at $\omega = 0$, from which it decreases smoothly to its first zero at $\omega = (2N+1)\pi/\Delta$. At larger frequency it oscillates with fixed period, and falls of as ω^{-2N}. The windowing function is finally given by:

$$W_N^{\Delta,T}(\omega) = \frac{2\sin(\omega(T-\Delta)/2)}{\omega} S_N^{\Delta}(\omega).$$

Without the shape function, this would represent the Fourier transform of the rectangular window given, in the time domain, by:

$$w(t) = \begin{cases} 1, & |t| < (T-\Delta)/2; \\ 0, & |t| > (T-\Delta)/2. \end{cases}$$

Whereas the Fourier transform of $w(t)$ falls off as ω^{-1} at large frequency, in the frequency domain the window function $W_N^{\Delta,T}(\omega)$ falls off as $\omega^{-(2N+1)}$.

The advantage of these windowing functions is obvious. Energy in narrow lines, or even from high average amplitude in a localized range of frequencies, is curtailed from feeding to other frequencies by the rapid fall-off rate outside the interval $(2N+1)\pi/\Delta$.

In the time domain, these window functions are defined as follows. Outside the interval $|t| < T/2$ they are zero exactly, while inside the interval $|t| < T/2 - \Delta$ they are exactly unity. In the intervening region they are defined in terms of an interpolating function. Let

$$J_N^{\Delta}(\tau) = \int_{\tau}^{\Delta} \sin(\pi x/\Delta)^{2N-1}\,dx, \text{ for } 0 < \tau < \Delta,$$

then

$$I_N^{\Delta,T}(t) = \frac{J_N^{\Delta}(t - T/2 + \Delta)}{J_N^{\Delta}(0)}, \text{ for } T/2 - \Delta < t < T/2$$

interpolates from one to zero, and gives rise to a globally C^{2N-1} window function in the time domain. The familiar Hann window function is an example of this construction, with $N = 1$ and $\Delta = T/2$. In this special case, there is no flat middle region, which arises only for $\Delta < T/2$. Larger N gives higher differentiability in the time domain, and larger fall-off in the frequency domain. While N and Δ can be chosen to control the window function, it should be noted that $\Delta < T/2$ will lead to less signal being "thrown away," and it will also delay the onset of the rapid fall-off.

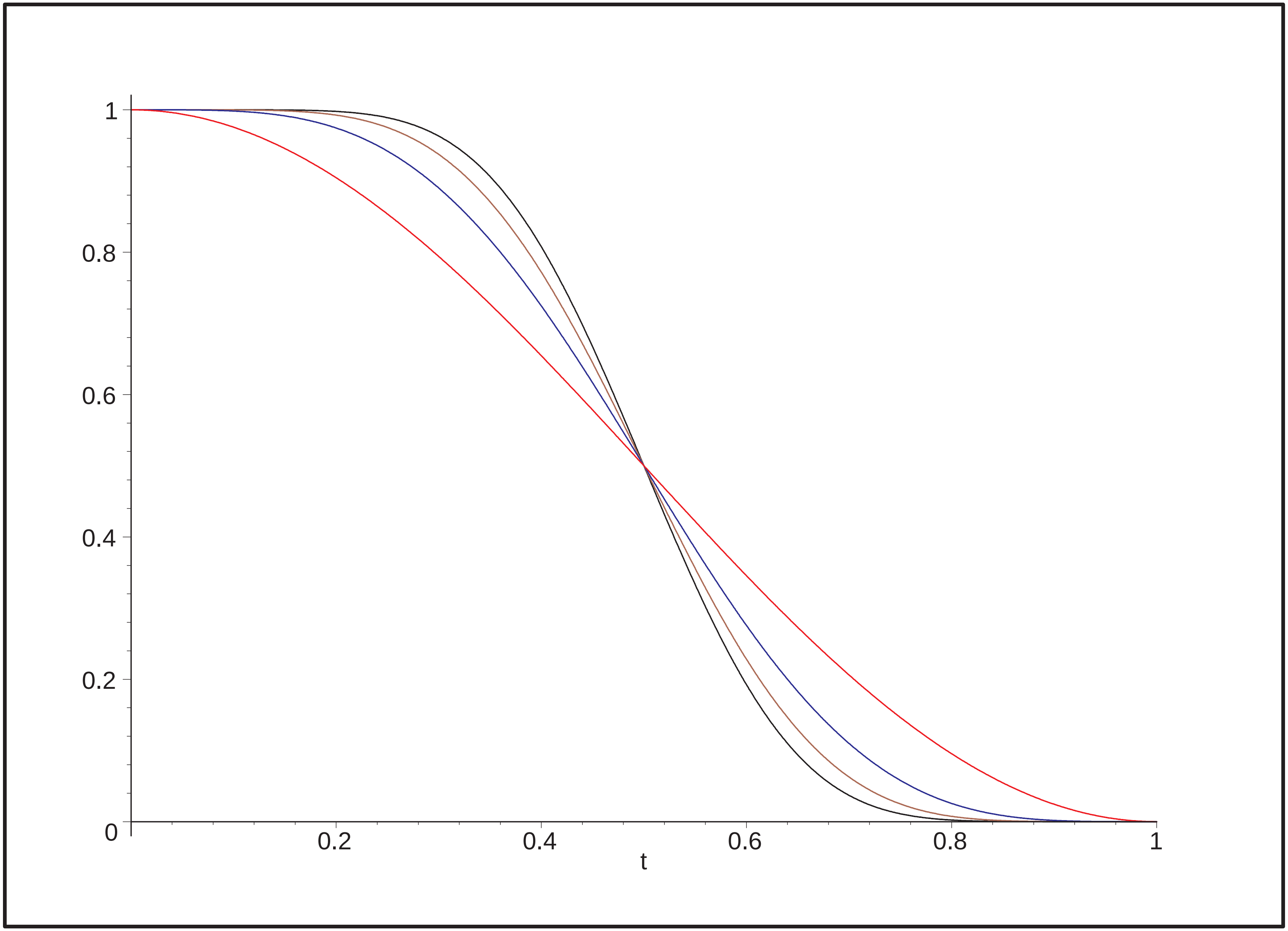
Application in LIGO Data Analysis

Three plots are included in the following pages for illustrative purposes.

Figure 1: The interpolating functions $I_N^{\Delta,T}$ for $N = 1-4$, $\Delta = 1$ & $T = 2$. Both the smoothness and the maximum steepness increase as N increases.

Figure 2: Log-log plot showing a comparison of the rectangular window fall-off (black) with that for several of the $W_N^{\Delta,T}$, specifically, those with $N = 1-3$, $\Delta = 1$ & $T = 2$. For comparison with discretely sampled data, the horizontal axis has been rescaled so that zero ($= \log(1)$) corresponds to exactly one frequency bin (say, $\delta f = 1/T$) no matter what the length of the interval sampled, nor whatever the sampling rate for the data. **Note:** For $N = 1$, and whenever $\Delta/T = 0.5$, we recover the familiar Hann window (here shown in red).

Figure 3: Log-log plot showing the delayed fall-off of one particular windowing function, $W_3^{\Delta,T}$, (which has eventual fall-off of f^{-7}) as Δ/T is decreased from 0.5 to 0.05. The rectangular window fall-off (black) remains displayed for comparison. Again the frequency axis has been scaled so that zero ($= \log(1)$) corresponds to exactly one frequency bin (*i.e.*, $\delta f \equiv 1/T$). In general, the rapid fall-off is delayed by a number of frequency bins which is proportional to TN/Δ.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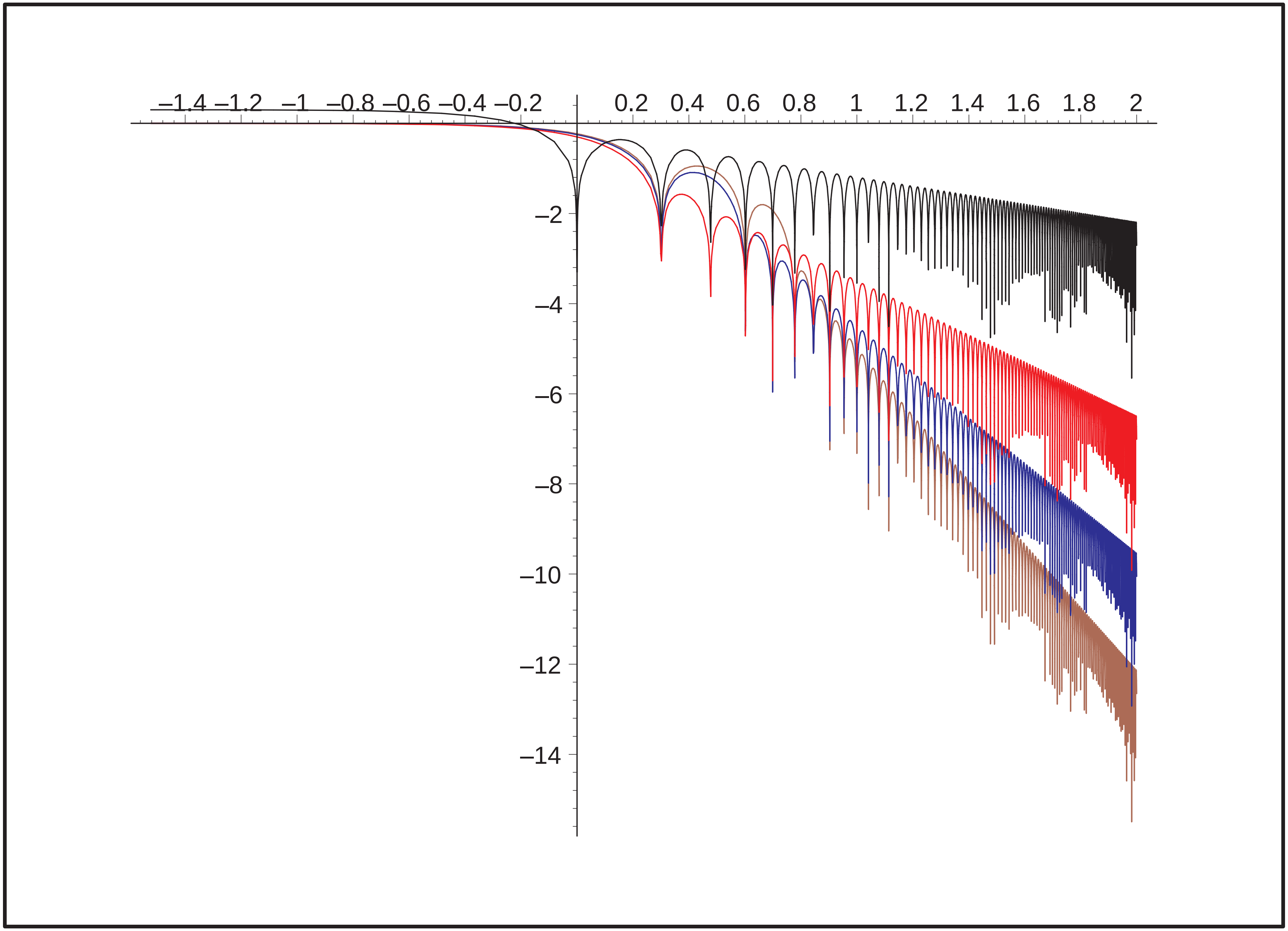

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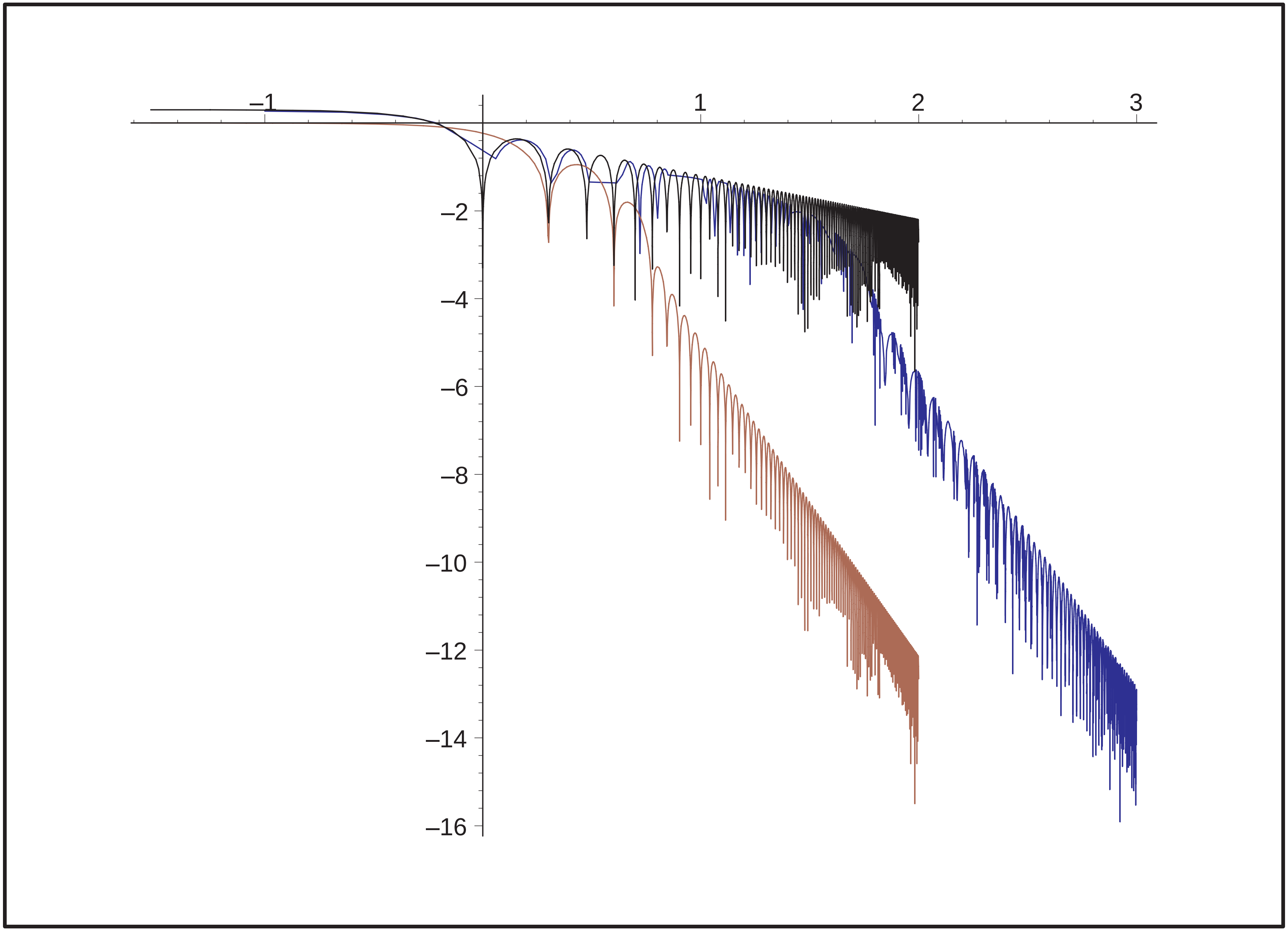

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