Gravity: Newtonian, post-Newtonian, Relativistic
Exercises

Newtonian Gravity

1. In post-Newtonian theory, there appears a "superpotential" X defined by

$$X(t,\boldsymbol{x}) = G\int \rho(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|d^3x' \,.$$

Show that $\nabla^2 X = 2U$, and that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}X(t,\boldsymbol{x}) &= -G\int \rho'\frac{d\boldsymbol{v}'}{dt}\cdot\frac{(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x' \\ &\quad + G\int \frac{\rho'}{|\boldsymbol{x}-\boldsymbol{x}'|}\left\{v'^2 - \frac{[\boldsymbol{v}'\cdot(\boldsymbol{x}-\boldsymbol{x}')]^2}{|\boldsymbol{x}-\boldsymbol{x}'|^2}\right\}d^3x' \,.\end{aligned}$$

2. Use the spherical-harmonic expansion of $|\boldsymbol{x}-\boldsymbol{x}'|^{-1}$ to verify that

$$U(t,r) = \frac{Gm(t,r)}{r} + 4\pi G\int_r^R \rho(t,r')r'dr'$$

for a spherical matter distribution.

3. Show explicitly that $\partial_{jknp}r^{-1} = 105n^{\langle jknp\rangle}/r^5$. Find $n^{\langle jknpq\rangle}$ by explicit construction.

4. Verify that the general N-body equation of motion for bodies with arbitrary multipole moments,

$$\begin{aligned}a^j_A = G\sum_{B\neq A}\Bigg\{&-\frac{m_B}{r^2_{AB}}n^j_{AB} + \sum_{\ell=2}^{\infty}\frac{1}{\ell!}\left[(-1)^\ell I^{\langle L\rangle}_B + \frac{m_B}{m_A}I^{\langle L\rangle}_A\right]\partial^A_{jL}\left(\frac{1}{r_{AB}}\right) \\ &+ \frac{1}{m_A}\sum_{\ell=2}^{\infty}\sum_{\ell'=2}^{\infty}\frac{(-1)^{\ell'}}{\ell!\ell'!}I^{\langle L\rangle}_A I^{\langle L'\rangle}_B\partial^A_{jLL'}\left(\frac{1}{r_{AB}}\right)\Bigg\}\,,\end{aligned}$$

satisfies $\sum_A m_A\boldsymbol{a}_A = 0$.

5. Suppose that the solar system is filled with a uniform distribution of dark matter with constant mass density ρ. Taking this distribution into account, calculate the modified gravitational potential of the Sun, and find the perturbing force $\boldsymbol{f}$ acting on a planetary orbit. Find the relation between orbital period P and semi-major axis a for a circular orbit, and calculate the secular changes in the planet's orbital elements. Place a bound on ρ using suitable solar-system data.

6. A test body of mass μ orbits a body of mass m, radius R, and dimensionless quadrupole moment J_2 relative to a symmetry axis $\boldsymbol{e}$; all other J_ℓ's are assumed to vanish. Prove that the following quantities are constants of the orbital motion:

   (a) The total energy, given by
$$E = \frac{1}{2}\mu v^2 - \frac{G\mu m}{r} + \frac{1}{2}\frac{G\mu m J_2 R^2}{r^3}\big[3(\boldsymbol{n}\cdot\boldsymbol{e})^2 - 1\big].$$

   (b) The angular momentum along $\boldsymbol{e}$, given by $L_e = \mu \boldsymbol{h}\cdot\boldsymbol{e}$, where $\boldsymbol{h} := \boldsymbol{r}\times\boldsymbol{v}$.

   (c) A third quantity, constant to first order in J_2, given by
$$C = h^2 - J_2 R^2\left[(\boldsymbol{e}\cdot\boldsymbol{v})^2 - 2\frac{Gm}{r}(\boldsymbol{e}\cdot\boldsymbol{n})^2\right],$$
   where $\boldsymbol{n} := \boldsymbol{r}/r$. This third constant is analogous to the "Carter constant" in the Kerr geometry of a rotating black hole.

7. Consider a spherical body on an inclined, circular orbit about an axisymmetric body of radius R and even multipole moments J_ℓ, with $\ell = 2, 4, 6$, and so on. To first order in perturbation theory, calculate the secular changes in the relevant orbital elements. In particular, show that:

   (a) the inclination is constant, that is, $\Delta\iota = 0$;

   (b) the line of nodes changes by an amount
$$\Delta\Omega = -3\pi\cos\iota\sum_{\ell=2}^{\infty} J_\ell\left(\frac{R}{p}\right)^\ell C_\ell,$$
   where $C_2 = 1$, $C_4 = -\frac{5}{2}(1 - \frac{7}{4}\sin^2\iota)$, and $C_6 = \frac{35}{8}(1 - \frac{9}{2}\sin^2\iota + \frac{33}{8}\sin^4\iota)$.

8. From the equations for Δe, $\Delta\omega$ and $\Delta\iota$ in the Kozai mechanism,
$$\langle\Delta e\rangle = \frac{15\pi}{2}\frac{m_3}{m}\left(\frac{a}{R}\right)^3 e(1-e^2)^{1/2}\sin^2\iota\sin\omega\cos\omega\,,$$
$$\langle\Delta\omega\rangle = \frac{3\pi}{2}\frac{m_3}{m}\left(\frac{a}{R}\right)^3(1-e^2)^{-1/2}\Big[5\cos^2\iota\sin^2\omega + (1-e^2)(5\cos^2\omega - 3)\Big],$$
$$\langle\Delta\iota\rangle = -\frac{15\pi}{2}\frac{m_3}{m}\left(\frac{a}{R}\right)^3 e^2(1-e^2)^{-1/2}\sin\iota\cos\iota\sin\omega\cos\omega\,,$$
   show that
$$e^2\cos^2\omega\sin^2\iota - \frac{3}{5}e^2 + \cos^2\iota = \text{constant}\,.$$

9. *Advanced problem.* Consider a point at a position $\bar{\boldsymbol{r}}(t)$ on a circular orbit of radius r around a central body of mass m, orbiting with angular velocity Ω, with $\Omega^2 = Gm/r^3$. Consider also a test body moving on nearby orbit, at a position $\delta\bar{\boldsymbol{r}}(t)$ relative to the point on the circular orbit. Assume that $\delta r \ll r$.

(a) In a coordinate system that rotates around the central body with angular velocity Ω, show that the equations of motion of the test body are given by

$$\frac{d^2}{dt^2}\delta\boldsymbol{r} + 2\boldsymbol{\Omega} \times \frac{d}{dt}\delta\boldsymbol{r} = \Omega^2 \Big[3(\boldsymbol{n}\cdot\delta\boldsymbol{r})\boldsymbol{n} - (\boldsymbol{e}_z \cdot \delta\boldsymbol{r})\boldsymbol{e}_z \Big]$$

to first order in $\delta\boldsymbol{r}$; here $\boldsymbol{\Omega} := \Omega\boldsymbol{e}_z$ is the angular-velocity vector, and $\boldsymbol{n} := \boldsymbol{r}/r$.

(b) Prove that the general solution to the equations of motion takes the form of the linear superposition $\delta\boldsymbol{r} = c_1\delta\boldsymbol{r}_1 + c_2\delta\boldsymbol{r}_2 + c_3\delta\boldsymbol{r}_3 + c_4\delta\boldsymbol{r}_4$, where c_n are arbitrary constants, and

$$\begin{aligned}
\delta\boldsymbol{r}_1 &= \cos(\Omega t - \chi_1)\,\boldsymbol{n} - 2\sin(\Omega t - \chi_1)\,\boldsymbol{\lambda}, \\
\delta\boldsymbol{r}_2 &= \boldsymbol{n} - \frac{3}{2}\Omega t\,\boldsymbol{\lambda}, \\
\delta\boldsymbol{r}_3 &= \boldsymbol{\lambda}, \\
\delta\boldsymbol{r}_4 &= \cos(\Omega t - \chi_4)\,\boldsymbol{e}_z
\end{aligned}$$

are the four eigenmodes of the perturbed orbit; χ_n are arbitrary phases.

(c) Describe the motion that corresponds to each mode, and show that each mode is generated by a perturbation in the orbital elements (p, e, ι, Ω) relative to the unperturbed, circular orbit. Relate the constants c_n to the variations of the orbital elements.

(d) Find a solution with $c_2 = c_3 = 0$, but with $c_1 \neq 0$ and $c_4 \neq 0$, describing a relative orbit that is circular, with a constant radius δr. What is the angle between the plane of the relative orbit and that of the original, unperturbed orbit?

(e) Now find a solution describing three satellites that are moving on the same circular relative orbit, such that initially they are placed at the vertices of an equilateral triangle. Show that as each satellite follows its orbit, the constellation maintains the shape of an equilateral triangle. This configuration was adopted for the three satellites making up the Laser Interferometer Space Antenna (LISA), a proposed space-based gravitational-wave detector.

Post-Minkowskian theory: Formulation

1. Show that $g_{\alpha\beta} = \sqrt{-\mathfrak{g}}\,\mathfrak{g}_{\alpha\beta}$, where $\mathfrak{g}_{\alpha\beta}$ is the matrix inverse to $\mathfrak{g}^{\alpha\beta}$, and $\mathfrak{g} = \det[\mathfrak{g}^{\alpha\beta}] = g$. If we define $\mathfrak{g}^{\alpha\beta} := \eta^{\alpha\beta} - h^{\alpha\beta}$, and $h^{\alpha\beta}$ is of order G, show that

$$\begin{aligned}
(-g) &= 1 - h + \frac{1}{2}h^2 - \frac{1}{2}h^{\mu\nu}h_{\mu\nu} + O(G^3)\,, \\
g_{\alpha\beta} &= \eta_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{2}h\eta_{\alpha\beta} + h_{\alpha\mu}h^{\mu}{}_{\beta} - \frac{1}{2}hh_{\alpha\beta} \\
&\quad + \left(\frac{1}{8}h^2 - \frac{1}{4}h^{\mu\nu}h_{\mu\nu}\right)\eta_{\alpha\beta} + O(G^3)\,,
\end{aligned}$$

where indices on $h^{\alpha\beta}$ are lowered and contracted with the Minkowski metric.

2. Consider the Schwarzschild metric in harmonic coordinates, given by

$$
\begin{aligned}
g_{00} &= -\frac{1 - R/2r_{\rm h}}{1 + R/2r_{\rm h}}, \\
g_{jk} &= \left(\frac{1 + R/2r_{\rm h}}{1 - R/2r_{\rm h}}\right) n_j n_k + \big(1 + R/2r_{\rm h}\big)^2 \big(\delta_{jk} - n_j n_k\big),
\end{aligned}
$$

where $n^j := x^j/r_{\rm h}$ is a radial unit vector, whose index is lowered with the Euclidean metric δ_{jk}, so that $n_j := \delta_{jk} n^k$. Show explicitly that

$$
\begin{aligned}
\mathfrak{g}^{00} &= -\frac{(1 + R/2r)^3}{1 - R/2r}\,, \\
\mathfrak{g}^{jk} &= \delta^{jk} - \left(\frac{R}{2r}\right)^2 n^j n^k\,,
\end{aligned}
$$

where $R := 2GM/c^2$, and verify that the harmonic gauge condition $\partial_\beta \mathfrak{g}^{\alpha\beta} = 0$ is satisfied.

3. Consider the potentials $h^{\alpha\beta}$ for a stationary source ($\partial_0 h^{\alpha\beta} = 0$), in harmonic gauge. Recalling Einstein's equations in the Landau-Lifshitz form:

$$
\partial_{\mu\nu} H^{\alpha\mu\beta\nu} = \frac{16\pi G}{c^4}(-g)\big(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL}\big)\,,
$$

along with the definitions $H^{\alpha\mu\beta\nu} = \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu}$, show that the conserved quantities M and P^j for the spacetime can be written in terms of the following surface integrals at infinity:

$$
\begin{aligned}
M &= -\frac{c^2}{16\pi G}\oint_\infty r^2 \frac{\partial h^{00}}{\partial r} d\Omega\,, \\
P^j &= -\frac{c^3}{16\pi G}\oint_\infty r^2 \frac{\partial h^{0j}}{\partial r} d\Omega\,,
\end{aligned}
$$

where $d\Omega = \sin\theta\, d\theta d\phi$ is the element of solid angle.

4. Given that $\tau^{\alpha\beta}$ is the source of the relaxed Einstein equation $\Box h^{\alpha\beta} = (16\pi G/c^4)\tau^{\alpha\beta}$ satisfying $\partial_\beta \tau^{\alpha\beta} = 0$, verify the identities

$$
\begin{aligned}
\tau^{0j} &= \partial_0\big(\tau^{00} x^j\big) + \partial_k\big(\tau^{0k} x^j\big)\,, \\
\tau^{jk} &= \frac{1}{2}\partial_{00}\big(\tau^{00} x^j x^k\big) + \frac{1}{2}\partial_p\big(2\tau^{p(j}x^{k)} - \partial_q \tau^{pq} x^j x^k\big)\,, \\
\tau^{0j} x^k &= \frac{1}{2}\partial_0\big(\tau^{00} x^j x^k\big) + \tau^{0[j}x^{k]} + \partial_p\big(\tau^{0p} x^j x^k\big)\,, \\
\tau^{jk} x^n &= \frac{1}{2}\partial_0\big(2\tau^{0(j}x^{k)}x^n - \tau^{0n} x^j x^k\big) \\
&\quad + \frac{1}{2}\partial_p\big(2\tau^{p(j}x^{k)}x^n - \tau^{np} x^j x^k\big)\,, \qquad (0.2)
\end{aligned}
$$

Using these identities verify that the near-zone expansion

$$
h^{00}_{\mathscr{N}}(t,\boldsymbol{x}) = \frac{4G}{c^4}\sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!\, c^\ell}\left(\frac{\partial}{\partial t}\right)^\ell \int_{\mathscr{M}} \tau^{00}(t,\boldsymbol{x'})|\boldsymbol{x} - \boldsymbol{x'}|^{\ell-1}\, d^3x'
$$

takes the form

$$\begin{aligned}
h^{00}_{\mathscr{N}} = \frac{4G}{c^2}\bigg\{ & \int_{\mathscr{M}} \frac{c^{-2}\tau^{00}}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x' + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int_{\mathscr{M}} c^{-2}\tau^{00}|\boldsymbol{x}-\boldsymbol{x'}|\,d^3x' \\
& - \frac{1}{6c^3}\overset{(3)}{\mathcal{I}}{}^{kk}(t) + \frac{1}{24c^4}\frac{\partial^4}{\partial t^4}\int_{\mathscr{M}} c^{-2}\tau^{00}|\boldsymbol{x}-\boldsymbol{x'}|^3\,d^3x' \\
& - \frac{1}{120c^5}\Big[(4x^kx^l + 2r^2\delta^{kl})\overset{(5)}{\mathcal{I}}{}^{kl}(t) - 4x^k\overset{(5)}{\mathcal{I}}{}^{kll}(t) + \overset{(5)}{\mathcal{I}}{}^{kkll}(t)\Big] \\
& + O(c^{-6})\bigg\} + h^{00}[\partial\mathscr{M}]\,,
\end{aligned}$$

modulo surface integrals denoted by $h^{00}[\partial\mathscr{M}]$, where $\mathcal{I}^L(t) := \int_{\mathscr{N}} \tau^{00}x^L d^3x$ and the symbol (n) on top of $\mathcal{I}$ denotes the number of time derivatives.

5. *Advanced problem.* This problem explores how to solve the Landau-Lifshitz formulation of the Einstein field equations for the Schwarzschild geometry.

   (a) Assuming static spherical symmetry, show that the general form of the gothic inverse metric in Cartesian coordinates can be written in the form

   $$\begin{aligned}
   \mathfrak{g}^{00} &= N(r)\,, \\
   \mathfrak{g}^{0j} &= 0\,, \\
   \mathfrak{g}^{jk} &= \alpha(r)P^{jk} + \beta(r)n^jn^k\,,
   \end{aligned}$$

   where N, α and β are arbitrary functions of r, n^j is a radial unit vector, and $P^{jk} := \delta^{jk} - n^jn^k$.

   (b) Show that $\mathfrak{g}_{\alpha\beta}$ is given by $\mathfrak{g}_{00} = N^{-1}$, $\mathfrak{g}_{jk} = \alpha^{-1}P^{jk} + \beta^{-1}n^jn^k$, and that $\mathfrak{g} := \det[\mathfrak{g}^{\alpha\beta}] = N\alpha^2\beta$.

   (c) Show that the imposition of the harmonic gauge condition leads to the constraint

   $$\beta' = \frac{2}{r}(\alpha - \beta)\,,$$

   where a prime indicates differentiation with respect to r. Recall that $\partial^j F(r) = F'(r)n^j$, and $\partial^j n^k = r^{-1}P^{jk}$.

   (d) Show that the three field equations that arise from the vacuum wave equation $\Box\mathfrak{g}^{\alpha\beta} = (16\pi G/c^4)\tau^{\alpha\beta}$ in harmonic coordinates have the form

   $$\begin{aligned}
   X' + XY + \frac{1}{r}(2X - Y) &= Q\,, \\
   XY + \frac{1}{r}(2X + Y) &= -Q\,, \\
   Z' + YZ + \frac{2}{r}Z &= Q\,,
   \end{aligned}$$

   where

   $$X := \frac{\alpha'}{\alpha}\,, \quad Y := \frac{\beta'}{\beta}\,, \quad Z := \frac{N'}{N}\,,$$

   and

   $$Q := \frac{1}{8}\Big(3Y^2 - Z^2 + 2YZ + 4XZ - 4XY\Big)\,.$$

Hint: One equation comes from the 00 component of the field equations, the other two come from splitting the jk components into a piece proportional to $n^j n^k$ and another piece proportional to P^{jk}. Use the gauge condition to simplify your expressions.

(e) By combining the first two field equations, obtain the solutions

$$X = 0 \quad \text{or} \quad r^4 \beta^2 X = c\,,$$

where $c \neq 0$ is a constant.

(f) Choosing the solution $X = 0$, show that the solutions for α and β that satisfy appropriate asymptotic conditions at $r = \infty$ are

$$\alpha = 1\,, \quad \beta = 1 - \frac{a}{r^2}\,,$$

where a is an arbitrary constant. Find the solution for N, determine a, and verify that the result is the Schwarzschild metric in harmonic coordinates.

(g) What is your interpretation of the second class of solutions, represented by a non-zero value of c? Show that by combining the equation $r^4\beta^2 X = c$ with the gauge condition, you can eliminate α and obtain the following differential equation for β:

$$W'' - \frac{W'}{r} = c\frac{W'}{W^2}\,,$$

where $W := r^2\beta$. Spend some time (but not too much!) trying to find a closed form solution to this nonlinear equation. (If you find one, please send it to us!)

Post-Newtonian theory: Near zone

1. The post-Newtonian metric is given by

$$\begin{aligned}
g_{00} &= -1 + \frac{2}{c^2}U + \frac{2}{c^4}\left(\Psi - U^2\right) + O(c^{-6})\,,\\
g_{0j} &= -\frac{4}{c^3}U_j + O(c^{-5})\,,\\
g_{jk} &= \delta_{jk}\left(1 + \frac{2}{c^2}U\right) + O(c^{-4})\,,
\end{aligned}$$

where

$$\Psi := \psi + \frac{1}{2}\partial_{tt}X\,.$$

and where the potentials are defined by

$$
\begin{aligned}
U(t,\boldsymbol{x}) &:= G\int \frac{\rho^{*\prime}}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x' \,, \\
\psi(t,\boldsymbol{x}) &:= G\int \frac{\rho^{*\prime}\bigl(\frac{3}{2}v'^2 - U' + \Pi' + 3p'/\rho^{*\prime}\bigr)}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x' \,, \\
X(t,\boldsymbol{x}) &:= G\int \rho^{*\prime}|\boldsymbol{x}-\boldsymbol{x'}|\, d^3x' \,, \\
U^j(t,\boldsymbol{x}) &:= G\int \frac{\rho^{*\prime}v'^j}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x' \,,
\end{aligned}
$$

Show that the inverse to the metric is given by

$$
\begin{aligned}
g^{00} &= -1 - \frac{2}{c^2}U - \frac{2}{c^4}\bigl(\Psi + U^2\bigr) + O(c^{-6}) \,, \\
g^{0j} &= -\frac{4}{c^3}U^j + O(c^{-5}) \,, \\
g^{jk} &= \left(1 - \frac{2}{c^2}U\right)\delta^{jk} + O(c^{-4}) \,,
\end{aligned}
$$

where $U^j := \delta^{jk}U_k$. Show that the metric determinant is $\sqrt{-g} = 1 + 2U/c^2 + O(c^{-4})$.

2. Show that, to the order necessary to obtain the equations of hydrodynamics $\nabla_\beta T^{\alpha\beta} = \partial_\beta T^{\alpha\beta} + \Gamma^\alpha_{\mu\beta}T^{\mu\beta} + \Gamma^\beta_{\mu\beta}T^{\alpha\mu} = 0$, the Christoffel symbols are given by

$$
\begin{aligned}
\Gamma^0_{00} &= -\frac{1}{c^3}\partial_t U + O(c^{-5}) \,, \\
\Gamma^0_{0j} &= -\frac{1}{c^2}\partial_j U + O(c^{-4}) \,, \\
\Gamma^0_{jk} &= \frac{2}{c^3}\bigl(\partial_j U_k + \partial_k U_j\bigr) + \frac{1}{c^3}\delta_{jk}\partial_t U + O(c^{-5}) \,, \\
\Gamma^j_{00} &= -\frac{1}{c^2}\partial_j U - \frac{1}{c^4}\bigl(4\partial_t U_j + \partial_j \Psi - 4U\partial_j U\bigr) + O(c^{-6}) \,, \\
\Gamma^j_{0k} &= \frac{1}{c^3}\delta_{jk}\partial_t U - \frac{2}{c^3}\bigl(\partial_k U_j - \partial_j U_k\bigr) + O(c^{-5}) \,, \\
\Gamma^j_{kn} &= \frac{1}{c^2}\bigl(\delta_{jn}\partial_k U + \delta_{jk}\partial_n U - \delta_{kn}\partial_j U\bigr) + O(c^{-4}) \,,
\end{aligned}
$$

3. Show that the N-body post-Newtonian equations of motion,

$$
\begin{aligned}
\boldsymbol{a}_A = &-\sum_{B\neq A}\frac{GM_B}{r_{AB}^2}\boldsymbol{n}_{AB} \\
&+\frac{1}{c^2}\Bigg(-\sum_{B\neq A}\frac{GM_B}{r_{AB}^2}\left[v_A^2-4(\boldsymbol{v}_A\cdot\boldsymbol{v}_B)+2v_B^2-\frac{3}{2}(\boldsymbol{n}_{AB}\cdot\boldsymbol{v}_B)^2\right. \\
&\qquad\qquad\qquad\qquad\left.-\frac{5GM_A}{r_{AB}}-\frac{4GM_B}{r_{AB}}\right]\boldsymbol{n}_{AB} \\
&+\sum_{B\neq A}\frac{GM_B}{r_{AB}^2}\Big[\boldsymbol{n}_{AB}\cdot(4\boldsymbol{v}_A-3\boldsymbol{v}_B)\Big](\boldsymbol{v}_A-\boldsymbol{v}_B) \\
&+\sum_{B\neq A}\sum_{C\neq A,B}\frac{G^2M_BM_C}{r_{AB}^2}\left[\frac{4}{r_{AC}}+\frac{1}{r_{BC}}-\frac{r_{AB}}{2r_{BC}^2}(\boldsymbol{n}_{AB}\cdot\boldsymbol{n}_{BC})\right]\boldsymbol{n}_{AB} \\
&-\frac{7}{2}\sum_{B\neq A}\sum_{C\neq A,B}\frac{G^2M_BM_C}{r_{AB}r_{BC}^2}\boldsymbol{n}_{BC}\Bigg)+O(c^{-4}),
\end{aligned}
$$

can be derived from the Lagrangian

$$
\begin{aligned}
L = &-\sum_A M_Ac^2\left[1-\frac{1}{2}(v_A/c)^2-\frac{1}{8}(v_A/c)^4\right]+\frac{1}{2}\sum_{A,B\neq A}\frac{GM_AM_B}{r_{AB}}\Bigg\{1 \\
&+\frac{1}{c^2}\left[3v_A^2-\frac{7}{2}\boldsymbol{v}_A\cdot\boldsymbol{v}_B-\frac{1}{2}(\boldsymbol{n}_{AB}\cdot\boldsymbol{v}_A)(\boldsymbol{n}_{AB}\cdot\boldsymbol{v}_B)-\sum_{C\neq A}\frac{GM_C}{r_{AC}}\right]\Bigg\}.
\end{aligned}
$$

4. Using the post-Newtonian two-body relative equation of motion

$$
\begin{aligned}
\boldsymbol{a} = &-\frac{Gm}{r^2}\boldsymbol{n}-\frac{Gm}{c^2r^2}\Bigg\{\left[(1+3\eta)v^2-\frac{3}{2}\eta\dot{r}^2-2(2+\eta)\frac{Gm}{r}\right]\boldsymbol{n} \\
&-2(2-\eta)\dot{r}\boldsymbol{v}\Bigg\}+O(c^{-4}),
\end{aligned}
$$

show that the post-Newtonian energy and angular momentum per unit reduced mass,

$$
\begin{aligned}
\varepsilon &:= \frac{1}{2}v^2-\frac{Gm}{r}+\frac{1}{c^2}\left\{\frac{3}{8}(1-3\eta)v^4+\frac{Gm}{2r}\left[(3+\eta)v^2+\eta\dot{r}^2+\frac{Gm}{r}\right]\right\} \\
&\quad +O(c^{-4}), \\
\boldsymbol{h} &:= \left\{1+\frac{1}{c^2}\left[\frac{1}{2}(1-3\eta)v^2+(3+\eta)\frac{Gm}{r}\right]\right\}(\boldsymbol{r}\times\boldsymbol{v})+O(c^{-4}),
\end{aligned}
$$

are conserved.

5. Consider the osculating equations for a two-body system in post-Newtonian

theory,

$$\begin{aligned}
\frac{dp}{df} &= 4(2-\eta)\frac{Gm}{c^2}e\sin f \\
\frac{de}{df} &= \frac{Gm}{c^2p}\biggl\{\Bigl[3-\eta+\frac{1}{8}(56-47\eta)e^2\Bigr]\sin f+(5-4\eta)e\sin 2f-\frac{3}{8}\eta e^2\sin 3f\biggr\}, \\
\frac{d\omega}{df} &= \frac{1}{e}\frac{Gm}{c^2p}\biggl\{3e-\Bigl[3-\eta-\frac{1}{8}(8+21\eta)e^2\Bigr]\cos f-(5-4\eta)e\cos 2f \\
&\quad +\frac{3}{8}\eta e^2\cos 3f\biggr\}.
\end{aligned}$$

(a) Obtain the net changes in each orbit element over one orbit.

(b) In the limit of small eccentricity. Show that a circular orbit (an orbit with r constant) does *not* correspond to $e=0$. Find a solution for p, e, and ω that corresponds to a circular post-Newtonian orbit, and give an interpretation of the orbit in the language of osculating Keplerian orbits.

6. Show that the magnification of the images of a Schwarzschild gravitational lens can be written in the form

$$\mu_\pm = \frac{1}{1-(\theta_{\rm E}/\theta_\pm)^4},$$

in which $\theta_{\rm E}$ is the Einstein angle and $\theta_\pm$ are the two solutions to the lens equation.

7. *Advanced problem.* Consider a Schwarzschild gravitational lens, and a circularly symmetric source whose center is at an undeflected angle β_0 from the lens in the x direction. Assume that the source has an angular diameter 2χ, with $\chi<\beta_0$, and model any point on the edge of the source as being on a circle described by $\beta(\phi)=(\beta_0+\chi\cos\phi)\boldsymbol{e}_x+\chi\sin\phi\,\boldsymbol{e}_y$, with ϕ ranging from 0 to 2π.

(a) In the limit $\chi\ll\beta_0$, show that the image is distorted into an ellipse, with a minor axis parallel to the direction of the image displacement, and with the ratio of minor to major axes given by $\beta_0/\sqrt{\beta_0^2+4\theta_{\rm E}^2}$.

(b) As χ increases for fixed β_0, show that the ellipse becomes concave, i.e. becomes an arc, when

$$\frac{\chi}{(\beta_0-\chi)\sqrt{\beta_0^2-\chi^2}} \ge \frac{1}{2\theta_{\rm E}}.$$

Post-Newtonian theory: Far zone

1. Consider a gravitational-wave field $h^{\alpha\beta}$ in the far-away wave zone, satisfying the harmonic gauge condition. Prove by direct calculation that

$$R_{0j0k} = -\frac{1}{2c^2}(\text{TT})^{jk}{}_{pq}\partial_{\tau\tau}h^{pq},$$

where the "transverse-traceless projector" $(\mathrm{TT})^{jk}{}_{pq}$ is given by

$$(\mathrm{TT})^{jk}{}_{pq} := P^j{}_p P^k{}_q - \frac{1}{2} P^{jk} P_{pq} \,,$$

where $P_{jk} \equiv \delta_{jk} - N_j N_k$.

2. An alternative way to study the polarizations of gravitational waves in the far-away wave zone is to focus on the Riemann tensor, and to exploit the fact that the waves, to lowest order in post-Minkowskian theory, propagate along null directions with respect to the background Minkowski spacetime. The idea, following Ted Newman and Roger Penrose, is to express the components of $R_{\alpha\beta\gamma\delta}$ on a basis of complex null vectors, defined by

$$\ell^\alpha := (1, \boldsymbol{N}) \,, \qquad n^\alpha := \frac{1}{2}(1, -\boldsymbol{N}) \,,$$

$$m^\alpha := \frac{1}{\sqrt{2}}(0, \boldsymbol{\vartheta} + \mathrm{i}\boldsymbol{\varphi}) \,, \qquad \bar{m}^\alpha := \frac{1}{\sqrt{2}}(0, \boldsymbol{\vartheta} - \mathrm{i}\boldsymbol{\varphi}) \,.$$

Here ℓ^α is an outgoing null vector tangent to the gravitational waves, n^α is an ingoing null vector, and the unit vectors $\boldsymbol{\vartheta}$ and $\boldsymbol{\varphi}$ in the directions transverse to the radial direction are defined by

$$\begin{aligned} \boldsymbol{\vartheta} &:= [\cos\vartheta\cos\varphi, \cos\vartheta\sin\varphi, -\sin\vartheta] \,, \\ \boldsymbol{\varphi} &:= [-\sin\varphi, \cos\varphi, 0] \,. \end{aligned}$$

Complex conjugation converts m^α to $\bar{m}^\alpha$ and vice versa.

(a) Prove the following properties of the basis vectors:

$$\ell_\alpha = -c\partial_\alpha(t - R/c) \,, \qquad n_\alpha = -\frac{c}{2}\partial_\alpha(t + R/c) \,,$$

$$\ell_\alpha \ell^\alpha = n_\alpha n^\alpha = m_\alpha m^\alpha = \bar{m}_\alpha \bar{m}^\alpha = 0 \,,$$

$$\ell_\alpha n^\alpha = -1 \,, \qquad m_\alpha \bar{m}^\alpha = 1 \,,$$

$$\eta^{\alpha\beta} = -2\ell^{(\alpha} n^{\beta)} + 2m^{(\alpha}\bar{m}^{\beta)} \,.$$

(b) Assume that the Riemann tensor in the far-away wave zone can be expressed as $R_{\alpha\beta\gamma\delta} = A_{\alpha\beta\gamma\delta}/R + O(R^{-2})$, in which $A_{\alpha\beta\gamma\delta}$ is an arbitrary function of retarded time $\tau := t - R/c$ and the unit vector $\boldsymbol{N}$. Show that

$$\partial_\mu R_{\alpha\beta\gamma\delta} = -\frac{1}{c}\ell_\mu \partial_\tau R_{\alpha\beta\gamma\delta} + O(R^{-2}) \,.$$

(c) Making use of this differentiation rule, use the linearized Bianchi identities

$$\partial_\epsilon R_{\alpha\beta\gamma\delta} + \partial_\delta R_{\alpha\beta\epsilon\gamma} + \partial_\gamma R_{\alpha\beta\delta\epsilon} = 0$$

to show that only the six components R_{npnq} can be nonzero, where the indices (p, q) run over the values ℓ, m, and $\bar{m}$. In this notation, for example, $R_{n\ell n\ell}$ stands for $R_{\alpha\beta\gamma\delta} n^\alpha \ell^\beta n^\gamma \ell^\delta$. You may ignore any constant of integration that arises when integrating with respect to retarded time.

(d) Calculate the Ricci tensor, and show that the vacuum Einstein field equations give rise to the four additional constraints

$$R_{n\ell n\ell} = R_{n\ell nm} = R_{n\ell n\bar{m}} = R_{nmn\bar{m}} = 0\,.$$

Show that there are only two unconstrained components, represented by R_{nmnm} and its complex conjugate (or equivalently, by its real and imaginary parts). These are the gravitational-wave modes, as represented by the Riemann tensor.

(e) Show that the link between the remaining components of the Riemann tensor and the gravitational-wave polarizations is provided by

$$R_{nmnm} = -\frac{1}{2c^2}\partial_{\tau\tau}h_{mm} = -\frac{1}{2c^2}\partial_{\tau\tau}(h_+ + \mathrm{i}h_\times)\,.$$

3. Consider an array of particles that are able to move freely in the x-y plane. A gravitational wave impinges on the plane in the z direction. It is described by polarizations h_+ and $h_\times$ defined relative to the x-y-z basis.

(a) Calculate the acceleration field $\ddot{\boldsymbol{\xi}}$ experienced by the particles. Draw the lines of force in the x-y plane when the wave is a pure $+$ polarization, and when it is a pure $\times$ polarization. How does the pattern change when the wave is a linear superposition of each polarization?

(b) Show that the local surface density of the particles is not affected by the gravitational wave, to first order in h_+ and $h_\times$. *Hint:* Evaluate the divergence of the displacement velocity field, $\boldsymbol{\nabla}\cdot\dot{\boldsymbol{\xi}}$.

(c) Show that the integral of the acceleration field around a closed path in the x-y plane always vanishes. Conclude that the acceleration field can be expressed as the gradient of a potential $\Phi_{\rm GW}$,

$$\ddot{\boldsymbol{\xi}} = \boldsymbol{\nabla}\Phi_{\rm GW}\,.$$

Determine $\Phi_{\rm GW}$ in terms of h_+ and $h_\times$.

4. The gravitational analogue of electromagnetic bremsstrahlung is a process in which a body of mass m_1 passes by a body of mass m_2 and is scattered by a small angle. This is the limit in which $v^2 \gg Gm/b$, where m is the total mass and b is the distance of closest approach. We still assume that $v \ll c$, and in this problem we employ the quadrupole formula to calculate the gravitational waves produced by the encounter.

The process corresponds to a Newtonian hyperbolic orbit with a very large eccentricity $e \gg 1$. (For $e > 1$ the semimajor axis a is not defined, but the semilatus rectum p is related as always to h, the angular momentum per unit reduced mass, by $h^2 = Gmp$.) We introduce the velocity at infinity defined by $v_\infty^2 := 2\varepsilon$, where ε is the conserved energy per unit reduced mass, and we define the impact parameter $b := p/e$.

(a) Using the Keplerian orbit formulae derived in earlier lectures, establish the following relations, assuming that the orbit is confined to the x-y plane, and that the orbit's pericenter is aligned with the x direction (so that $\omega = 0$):

$$\begin{aligned} v_\infty &= \sqrt{\frac{Gm}{p}}\, e\left[1 - \frac{1}{2}e^{-2} + O(e^{-4})\right], \\ r &= \frac{b}{\cos\phi}\left[1 - \frac{1}{e\cos\phi} + O(e^{-2})\right], \\ \boldsymbol{v} &= v_\infty\big[-e^{-1}\sin\phi, 1 + e^{-1}\cos\phi, 0\big] + O(e^{-2}). \end{aligned}$$

(b) Integrate the orbital equation for ϕ to leading order in e^{-1}, and show that

$$\sin\phi = \frac{v_\infty t}{(b^2 + v_\infty^2 t^2)^{1/2}} + O(e^{-1}), \qquad \cos\phi = \frac{b}{(b^2 + v_\infty^2 t^2)^{1/2}} + O(e^{-1}).$$

(c) Using the quadrupole formula, and taking the waves to be propagating in the direction of the vector $\boldsymbol{N} = [\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta]$, show that the gravitational-wave polarizations are given by

$$h_{+,\times} = \frac{2\eta(Gm)^2}{c^4 bR} A_{+,\times},$$

in which $\eta := m_1 m_2/m^2$ and

$$\begin{aligned} A_+ &= -\frac{1}{2}(1 + \cos^2\vartheta)\big[\cos 2\varphi\,(C_1 + 2C_3) + 2\sin 2\varphi\,(S_1 + S_3)\big] \\ &\quad - \frac{1}{2}\sin^2\vartheta\, C_1, \\ A_\times &= -\cos\vartheta\big[2\cos 2\varphi\,(S_1 + S_3) - \sin 2\varphi\,(C_1 + 2C_3)\big], \end{aligned}$$

where $C_n := \cos^n\phi$ and $S_n := \sin\phi\cos^{n-1}\phi$. An unobservable constant contribution to $h_{+,\times}$ has been dropped.

(d) Plot A_+ and $A_\times$ as a function of time in units of $t_0 = b/v_\infty$ for the following sets of directions (in degrees): $(\vartheta, \varphi) = (0, 0)$, $(45, 0)$, $(90, 0)$, $(90, 45)$, $(90, 90)$, $(45, 90)$, and $(60, 54.7)$, the last point corresponding to a direction in a plane tilted 45 degrees relative to the orbital plane, and 45 degrees from the y-direction in this plane. Running the plots from $t = -10t_0$ to $t = +10t_0$ will reveal the salient features.

(e) Some of the waveforms have an unusual feature. What is it? Discuss whether it might be observable to any practical gravitational wave detector.

5. The output of a laser interferometer like LIGO is governed by the signal $S(t)$, where

$$S(t) = \frac{1}{2}\big(e_1^j e_1^k - e_2^j e_2^k\big) A_{\rm TT}^{jk}(\tau, \boldsymbol{N}),$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are unit vectors pointing along the two arms. The transverse-traceless wave amplitude $A_{\rm TT}^{jk}$ is related to the $+$ and $\times$ polarization modes linked to the source by

$$A_{\rm TT}^{jk} = \big(e_X^j e_X^k - e_Y^j e_Y^k\big) A_+ + \big(e_X^j e_Y^k + e_Y^j e_X^k\big) A_\times,$$

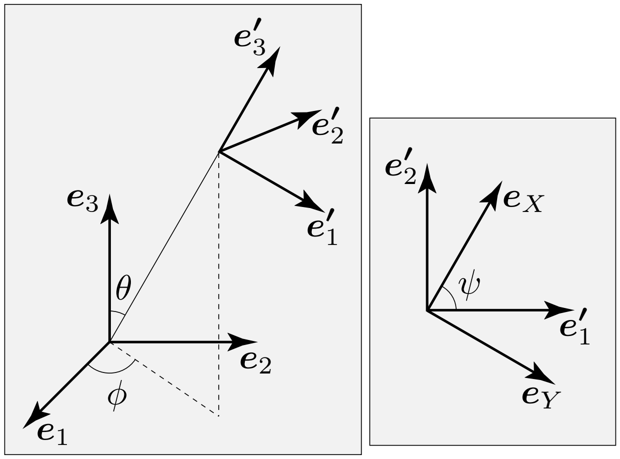

Figure 1: Relation between the detector basis $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ and the transverse basis $(\boldsymbol{e}_X, \boldsymbol{e}_Y)$.

where $\boldsymbol{e}_X$ and $\boldsymbol{e}_Y$ are unit vectors perpendicular to the propagation direction $\boldsymbol{N}$. The relationship between $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ and $\boldsymbol{e}_X$ and $\boldsymbol{e}_Y$ is given in the Figure, where θ and ϕ are the polar angles indicating the direction of the source as seen from the interferometer, and ψ is a polarization angle defining the orientation of the $X - Y$ basis about $\boldsymbol{N}$.

(a) Show that $S(t) = F_+ A_+ + F_\times A_\times$ where the detector pattern functions F_+ and $F_\times$ are given by

$$\begin{aligned} F_+ &= \frac{1}{2}(1+\cos^2\theta)\cos 2\phi \cos 2\psi - \cos\theta \sin 2\phi \sin 2\psi, \\ F_\times &= \frac{1}{2}(1+\cos^2\theta)\cos 2\phi \sin 2\psi + \cos\theta \sin 2\phi \cos 2\psi\,. \end{aligned}$$

(b) Show that the detector pattern functions for an interferometer whose arms make an angle χ with each other are the same as in part (a), but multiplied by $\sin\chi$. *Hint*: Orient the new arms in the 1-2 plane so that each one makes an angle $\frac{\pi}{4} - \frac{1}{2}\chi$ with respect to the $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ axes.

6. From the quadrupole formula for energy flux

$$\frac{dE}{dt} = \frac{G}{5c^5} \dddot{I}^{\langle pq \rangle} \dddot{I}^{\langle pq \rangle} \,,$$

the definition $I^{jk} \equiv \int \rho^* x^j x^k d^3x$, and the Newtonian equations of motion, show that the energy flux from a binary system is given by

$$\frac{dE}{dt} = \frac{8}{15}\eta^2 \frac{c^3}{G}\left(\frac{Gm}{c^2 r}\right)^4 \left(12v^2 - 11\dot{r}^2\right).$$

Averaging over a Newtonian eccentric orbit, show that the energy flux becomes

$$\langle \frac{dE}{dt} \rangle = \frac{32}{5}\eta^2 \frac{c^5}{G}\left(\frac{Gm}{c^2 a}\right)^5 (1-e^2)^{-7/2}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right).$$

Show that, as a consequence, the orbital period P decreases at a rate given by

$$\frac{dP}{dt} = -\frac{192\pi}{5}\left(\frac{G\mathcal{M}}{c^3}\frac{2\pi}{P}\right)^{5/3}\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}},$$

where $\mathcal{M} \equiv \eta^{3/5}m$ is the "chirp" mass.

7. Consider a Keplerian orbit that is circular apart from the slow decrease in radius a caused by the energy lost to gravitational radiation. As a function of η, m, and the initial radius a_0, calculate the lifetime of the binary system and the number of completed orbits before the radiation reaction brings the radius to zero. Give alternative expressions for the lifetime and number of orbits in terms of η, m, and the initial orbital period P. Using these results, carry out the following estimates:

   (a) the remaining lifetime of the Hulse-Taylor binary pulsar PSR 1913+16, with $M_1 \approx M_2 \approx 1.4M_\odot$ and $P = 7.75$ hours (assume that the orbit is circular);

   (b) the total time and number of cycles in the gravitational-wave signal from an inspiralling binary system of two $1.4M_\odot$ compact objects, from the time it enters the LIGO-Virgo frequency band with a gravitational-wave frequency of 10 Hz to the end of the inspiral (when $a = 0$);

   (c) the remaining lifetime of the Earth-Sun system.

8. The current eccentricity of the Hulse-Taylor binary pulsar orbit is $e_0 \approx 0.6$, and its orbital period is 7.75 hours. Estimate the orbital eccentricity when gravitational waves from the system first enter the LIGO-Virgo band at 10 Hz. You may treat the eccentricity as if it were much smaller than unity when making your estimate.