The two-body Kepler problem

- set center of mass at the origin (X = 0)
- ignore all multipole moments (spherical bodies or point masses)
- define $\boldsymbol{r} := \boldsymbol{r}_1 - \boldsymbol{r}_2 \,,\; r := |\boldsymbol{r}| \,,\; m := m_1 + m_2 \,,\; \mu := m_1 m_2 / m$
- reduces to effective one-body problem

$$\boldsymbol{a} = -\frac{Gm}{r^2}\boldsymbol{n}$$

Energy and angular momentum conserved:

$$\begin{aligned} E &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - G\frac{m_1 m_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \\ &= \frac{1}{2}\mu v^2 - G\frac{\mu m}{r} \end{aligned}$$

$$\begin{aligned} L &= m_1 \boldsymbol{r}_1 \times \boldsymbol{v}_1 + m_2 \boldsymbol{r}_2 \times \boldsymbol{v}_2 \\ &= \mu \boldsymbol{r} \times \boldsymbol{v} \end{aligned}$$

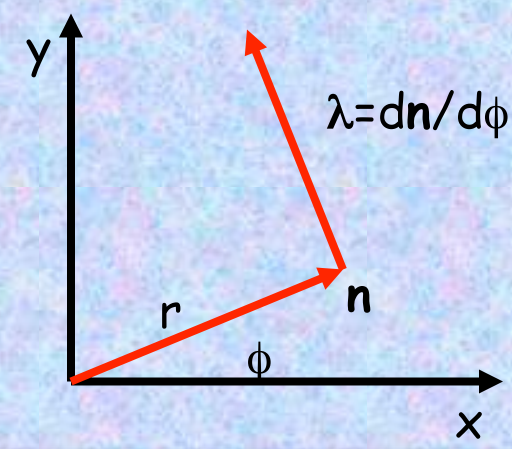
orbital plane is fixed

Effective one-body problem

Make orbital plane the x-y plane

$$\boldsymbol{r} \times \boldsymbol{v} = r^2 \frac{d\phi}{dt} := h\boldsymbol{e}_z$$

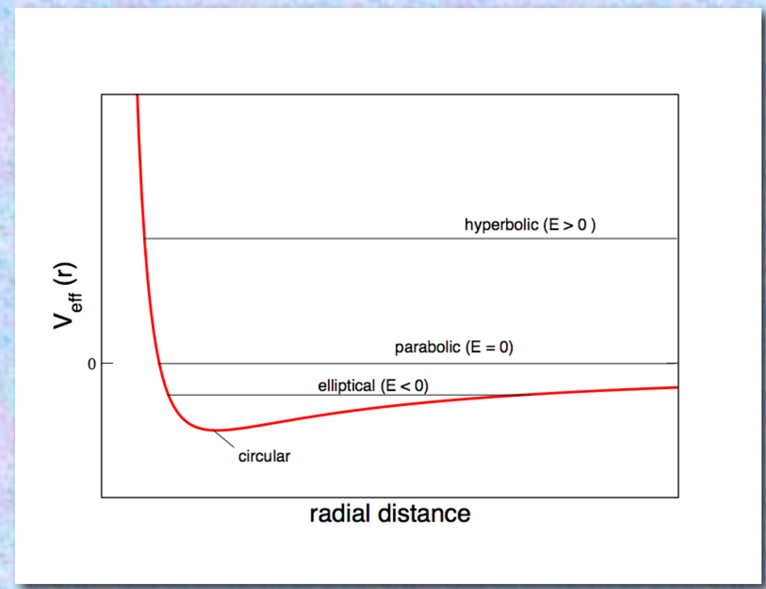
$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \dot{r}\boldsymbol{n} + r\dot{\phi}\boldsymbol{\lambda}$$



From energy conservation: $\varepsilon = E/\mu$

$$\dot{r}^2 = 2\left[\varepsilon - V_{\rm eff}(r)\right]$$

$$V_{\rm eff}(r) = \frac{h^2}{r^2} - \frac{Gm}{r}$$



Reduce to quadratures (integrals)

$$t - t_i = \pm \int_{r_i}^{r} \frac{dr'}{\sqrt{2[\varepsilon - V_{\rm eff}(r')]}}$$

$$\phi - \phi_i = h \int_{t_i}^{t} \frac{dt'}{r(t')^2}$$

Keplerian orbit solutions

Radial acceleration, or d/dt of energy equation:

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{Gm}{r^2}$$

Find the orbit in space: convert from t to ϕ:

$$d/dt = \dot{\phi} d/d\phi = (h/r^2) d/d\phi$$

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{Gm}{h^2}$$

$$\frac{1}{r} = \frac{1}{p}(1 + e\cos f)$$

$f := \phi - \omega$ true anomaly

$p := h^2/Gm$ semilatus rectum

Elliptical orbits (e < 1, a > 0)

$$r_{\text{peri}} = \frac{p}{1+e}\,, \quad \phi = \omega$$

$$r_{\text{apo}} = \frac{p}{1-e}\,, \quad \phi = \omega + \pi$$

$$a := \frac{1}{2}(r_{\text{peri}} + r_{\text{apo}}) = \frac{p}{1-e^2}$$

Hyperbolic orbits (e > 1, a < 0)

$$\phi_{\text{in}} - \phi_{\text{out}} = \pi - 2\arcsin(1/e)$$

Keplerian orbit solutions

Useful relationships

$$\dot{r} = \frac{he}{p} \sin f$$

$$v^2 = \frac{Gm}{p}(1 + 2e \cos f + e^2) = Gm \left(\frac{2}{r} - \frac{1}{a} \right)$$

$$E = -\frac{G\mu m}{2a}$$

$$e^2 = 1 + \frac{2h^2 E}{\mu (Gm)^2}$$

$$P = 2\pi \left(\frac{a^3}{Gm} \right)^{1/2}$$ for closed orbits

Alternative solution

$$r = a(1 - e \cos u)$$

$$n(t - T) = u - e \sin u$$

$$\tan \frac{f}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2}$$

$$n = 2\pi / P$$

u = eccentric anomaly

f = true anomaly

n = mean motion

Dynamical symmetry in the Kepler problem

- a and e are constant (related to E and h)
- orbital plane is constant (related to direction of h)
- ω is constant -- a hidden, dynamical symmetry

Runge-Lenz vector

$$\begin{aligned} \boldsymbol{A} &:= \frac{\boldsymbol{v} \times \boldsymbol{h}}{Gm} - \boldsymbol{n} \\ &= e\big(\cos\omega\, \boldsymbol{e}_x + \sin\omega\, \boldsymbol{e}_y\big) \\ &= \text{constant} \end{aligned}$$

Comments:

- responsible for the degeneracy of hydrogen energy levels
- added symmetry occurs only for 1/r and r^2 potentials
- deviation from 1/r potential generically causes $d\omega/dt$

Keplerian orbit in space

Six orbit elements:

- i = inclination relative to reference plane:

$$\cos\iota = \hat{\boldsymbol{h}} \cdot \boldsymbol{e}_Z$$

- Ω = angle of ascending node

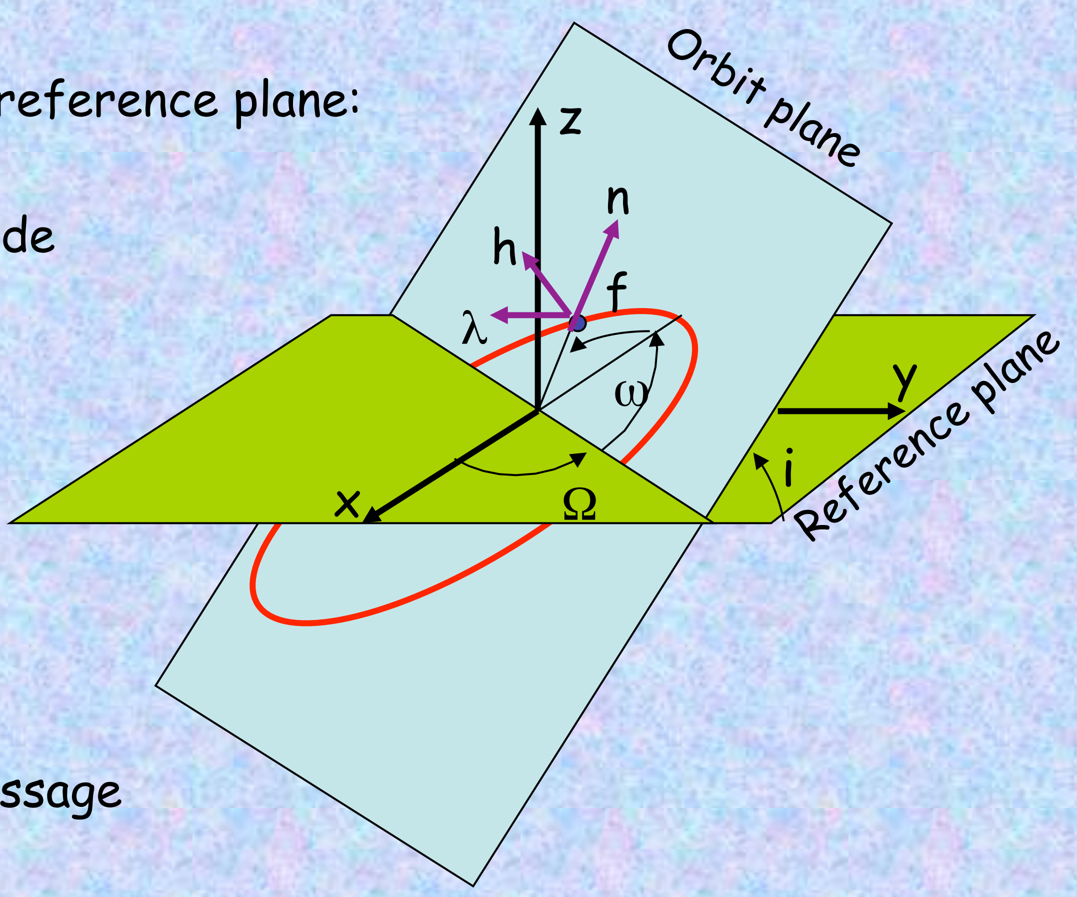
$$\cos\Omega = -\frac{\hat{\boldsymbol{h}} \cdot \boldsymbol{e}_Y}{\sin\iota}$$

- ω = angle of pericenter

$$\sin\omega = \frac{\boldsymbol{A} \cdot \boldsymbol{e}_z}{e \sin\iota}$$

- e = |**A**|
- a = h^2/Gm(1-e^2)
- T = time of pericenter passage

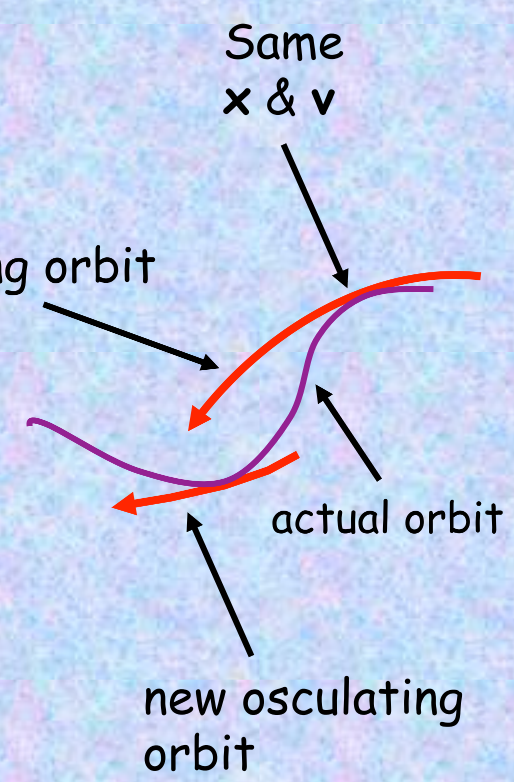
$$T = t - \int_0^f \frac{r^2}{h} df$$



Comment: equivalent to the initial conditions $\mathbf{x}_0$ and $\mathbf{v}_0$

Osculating orbit elements and the perturbed Kepler problem

$$\boldsymbol{a} = -\frac{Gm\boldsymbol{r}}{r^3} + \boldsymbol{f}(\boldsymbol{r}, \boldsymbol{v}, t)$$



Define:

$$\boldsymbol{r} := r\boldsymbol{n}\,, \quad r := p/(1+e\cos f)\,, \quad p = a(1-e^2)$$

$$\boldsymbol{v} := \frac{he\sin f}{p}\boldsymbol{n} + \frac{h}{r}\boldsymbol{\lambda}, \quad h := \sqrt{Gmp}$$

$$\begin{aligned}\boldsymbol{n} :=& \left[\cos\Omega\cos(\omega+f) - \cos\iota\sin\Omega\sin(\omega+f)\right]\boldsymbol{e}_X \\ &+ \left[\sin\Omega\cos(\omega+f) + \cos\iota\cos\Omega\sin(\omega+f)\right]\boldsymbol{e}_Y \\ &+ \sin\iota\sin(\omega+f)\,\boldsymbol{e}_Z\end{aligned}$$

$$\begin{aligned}\boldsymbol{\lambda} :=& \left[-\cos\Omega\sin(\omega+f) - \cos\iota\sin\Omega\cos(\omega+f)\right]\boldsymbol{e}_X \\ &+ \left[-\sin\Omega\sin(\omega+f) + \cos\iota\cos\Omega\cos(\omega+f)\right]\boldsymbol{e}_Y \\ &+ \sin\iota\cos(\omega+f)\,\boldsymbol{e}_Z\end{aligned}$$

$$\hat{\boldsymbol{h}} := \boldsymbol{n}\times\boldsymbol{\lambda} = \sin\iota\sin\Omega\,\boldsymbol{e}_X - \sin\iota\cos\Omega\,\boldsymbol{e}_Y + \cos\iota\,\boldsymbol{e}_Z$$

e, a, ω, Ω, i, T may be functions of time

Perturbed Kepler problem

$$\boldsymbol{a} = -\frac{Gm\boldsymbol{r}}{r^3} + \boldsymbol{f}(\boldsymbol{r}, \boldsymbol{v}, t)$$

$$\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v} \implies \frac{d\boldsymbol{h}}{dt} = \boldsymbol{r} \times \boldsymbol{f}$$

$$\boldsymbol{A} = \frac{\boldsymbol{v} \times \boldsymbol{h}}{Gm} - \boldsymbol{n} \implies Gm\frac{d\boldsymbol{A}}{dt} = \boldsymbol{f} \times \boldsymbol{h} + \boldsymbol{v} \times (\boldsymbol{r} \times \boldsymbol{f})$$

Decompose: $\boldsymbol{f} = \mathcal{R}\boldsymbol{n} + \mathcal{S}\boldsymbol{\lambda} + \mathcal{W}\hat{\boldsymbol{h}}$

$$\frac{d\boldsymbol{h}}{dt} = -r\mathcal{W}\,\boldsymbol{\lambda} + r\mathcal{S}\,\hat{\boldsymbol{h}}$$

$$Gm\frac{d\boldsymbol{A}}{dt} = 2h\mathcal{S}\,\boldsymbol{n} - \left(h\mathcal{R} + r\dot{r}\mathcal{S}\right)\boldsymbol{\lambda} - r\dot{r}\mathcal{W}\,\hat{\boldsymbol{h}}.$$

Example: $\dot{h} = r\mathcal{S}$

$$\frac{d}{dt}(h\cos\iota) = \dot{\boldsymbol{h}} \cdot \boldsymbol{e}_Z$$

$$\dot{h}\cos\iota - h\frac{d\iota}{dt}\sin\iota = -r\mathcal{W}\cos(\omega + f)\sin\iota + r\mathcal{S}\cos\iota$$

Perturbed Kepler problem

"Lagrange planetary equations"

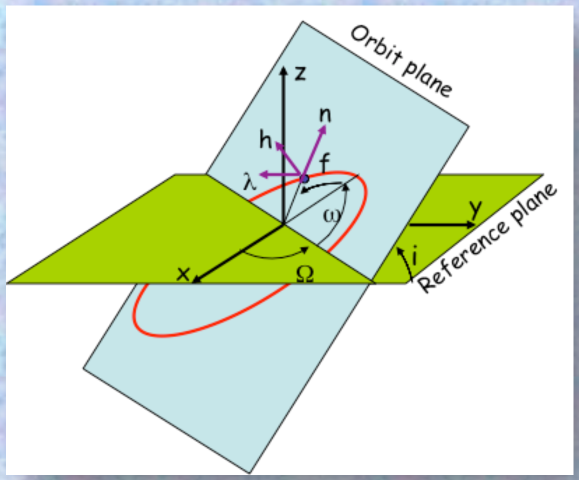
$$\frac{dp}{dt} = 2\sqrt{\frac{p^3}{Gm}}\frac{1}{1+e\cos f}\mathcal{S},$$

$$\frac{de}{dt} = \sqrt{\frac{p}{Gm}}\left[\sin f\,\mathcal{R} + \frac{2\cos f + e(1+\cos^2 f)}{1+e\cos f}\mathcal{S}\right],$$

$$\frac{d\iota}{dt} = \sqrt{\frac{p}{Gm}}\frac{\cos(\omega+f)}{1+e\cos f}\mathcal{W},$$

$$\sin\iota\frac{d\Omega}{dt} = \sqrt{\frac{p}{Gm}}\frac{\sin(\omega+f)}{1+e\cos f}\mathcal{W},$$

$$\frac{d\omega}{dt} = \frac{1}{e}\sqrt{\frac{p}{Gm}}\left[-\cos f\,\mathcal{R} + \frac{2+e\cos f}{1+e\cos f}\sin f\,\mathcal{S} - e\cot\iota\frac{\sin(\omega+f)}{1+e\cos f}\mathcal{W}\right]$$



An alternative pericenter angle:

$$\varpi := \omega + \Omega\cos\iota$$

$$\frac{d\varpi}{dt} = \frac{1}{e}\sqrt{\frac{p}{Gm}}\left[-\cos f\,\mathcal{R} + \frac{2+e\cos f}{1+e\cos f}\sin f\,\mathcal{S}\right]$$

Perturbed Kepler problem

Comments:

- these six 1st-order ODEs are exactly equivalent to the original three 2nd-order ODEs
- if **f** = 0, the orbit elements are constants
- if **f** << Gm/r^2, use perturbation theory
- yields both periodic and secular changes in orbit elements
- can convert from d/dt to d/df using

$$\frac{df}{dt} = \frac{h}{r^2} - \left(\frac{d\omega}{dt} + \cos\iota \frac{d\Omega}{dt} \right)$$

Drop if working to 1st order

"Secular" variations of orbit elements

$$\frac{dX_\alpha}{dt} = Q_\alpha(X_\beta, t)$$

$$\frac{df}{dt} = \frac{h}{r^2} = \frac{m^{1/2}}{p^{3/2}}(1 + e\cos f)^2$$

$$\Longrightarrow \frac{dX_\alpha}{df} = \tilde{Q}_\alpha(X_\beta, f)$$

First order perturbation theory: set the X_β = const in the RHS and integrate

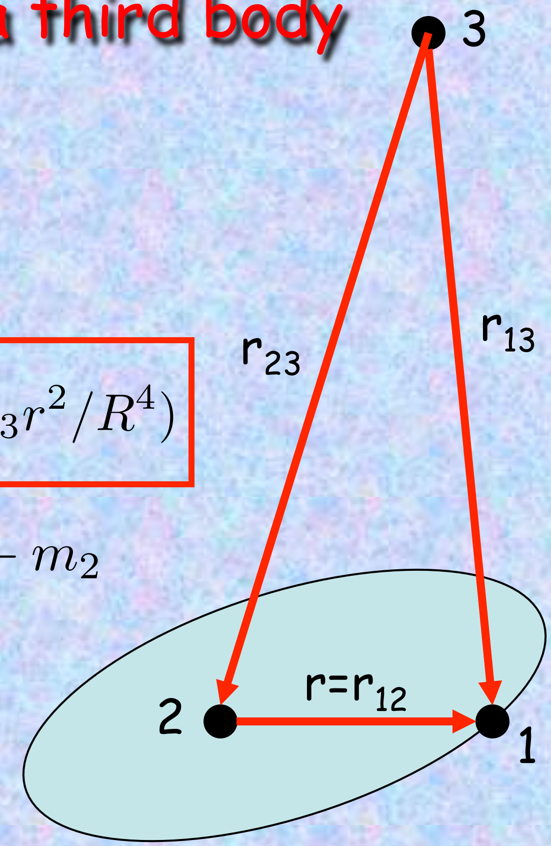
$$X_\alpha(f) = \int_0^f \tilde{Q}_\alpha(X_\beta, f')df'$$

There could be a net change in X_α over one orbit - called a "secular" effect

$$\Delta X_\alpha = \int_0^{2\pi} \tilde{Q}_\alpha(X_\beta, f')df'$$

Perturbed Kepler problem

Worked example: perturbations by a third body



$$\boldsymbol{a}_1 = -Gm_2 \frac{\boldsymbol{r}_{12}}{r_{12}^3} - Gm_3 \frac{\boldsymbol{r}_{13}}{r_{13}^3},$$

$$\boldsymbol{a}_2 = +Gm_1 \frac{\boldsymbol{r}_{12}}{r_{12}^3} - Gm_3 \frac{\boldsymbol{r}_{23}}{r_{23}^3}$$

$$\boxed{\boldsymbol{a} = \frac{Gm\boldsymbol{r}}{r^3} - \frac{Gm_3 r}{R^3}\Big[\boldsymbol{n} - 3(\boldsymbol{n}\cdot\boldsymbol{N})\boldsymbol{N}\Big] + O(Gm_3 r^2/R^4)}$$

$$R := |\boldsymbol{r}_{23}|\,,\; N := \boldsymbol{r}_{23}/|\boldsymbol{r}_{23}|\,,\; m := m_1 + m_2$$

$$\mathcal{R} := \boldsymbol{f}\cdot\boldsymbol{n} = -\frac{Gm_3 r}{R^3}\big[1 - 3(\boldsymbol{n}\cdot\boldsymbol{N})^2\big],$$

$$\mathcal{S} := \boldsymbol{f}\cdot\boldsymbol{\lambda} = 3\frac{Gm_3 r}{R^3}(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{\lambda}\cdot\boldsymbol{N}),$$

$$\mathcal{W} := \boldsymbol{f}\cdot\hat{\boldsymbol{h}} = 3\frac{Gm_3 r}{R^3}(\boldsymbol{n}\cdot\boldsymbol{N})(\hat{\boldsymbol{h}}\cdot\boldsymbol{N})$$

Put third body on a circular orbit

$$\boldsymbol{N} = \boldsymbol{e}_X \cos F + \boldsymbol{e}_Y \sin F\,, \qquad \frac{dF}{dt} = \sqrt{\frac{G(m+m_3)}{R^3}} \ll \frac{df}{dt}$$

Perturbed Kepler problem

Worked example: perturbations by a third body

Integrate over f from 0 to 2π holding F fixed, then average over F from 0 to 2π:

$$\langle \Delta a \rangle = 0$$

$$\langle \Delta e \rangle = \frac{15\pi}{2}\frac{m_3}{m}\left(\frac{a}{R}\right)^3 e(1-e^2)^{1/2}\sin^2\iota\sin\omega\cos\omega$$

$$\langle \Delta \omega \rangle = \frac{3\pi}{2}\frac{m_3}{m}\left(\frac{a}{R}\right)^3 (1-e^2)^{-1/2}\Big[5\cos^2\iota\sin^2\omega + (1-e^2)(5\cos^2\omega - 3)\Big]$$

$$\langle \Delta \iota \rangle = -\frac{15\pi}{2}\frac{m_3}{m}\left(\frac{a}{R}\right)^3 e^2(1-e^2)^{-1/2}\sin\iota\cos\iota\sin\omega\cos\omega$$

$$\langle \Delta \Omega \rangle = -\frac{3\pi}{2}\frac{m_3}{m}\left(\frac{a}{R}\right)^3 (1-e^2)^{-1/2}(1-5e^2\cos^2\omega + 4e^2)\cos\iota$$

Also:

$$\langle \Delta \varpi \rangle = \frac{3\pi}{2}\frac{m_3}{m}\left(\frac{a}{R}\right)^3 (1-e^2)^{1/2}\Big[1 + \sin^2\iota(1-5\sin^2\omega)\Big]$$

Perturbed Kepler problem

Worked example: perturbations by a third body

Case 1: coplanar 3rd body and Mercury's perihelion (i = 0)

$$\langle \Delta\varpi \rangle = \frac{3\pi}{2}\frac{m_3}{m}\left(\frac{a}{R}\right)^3 (1-e^2)^{1/2}$$

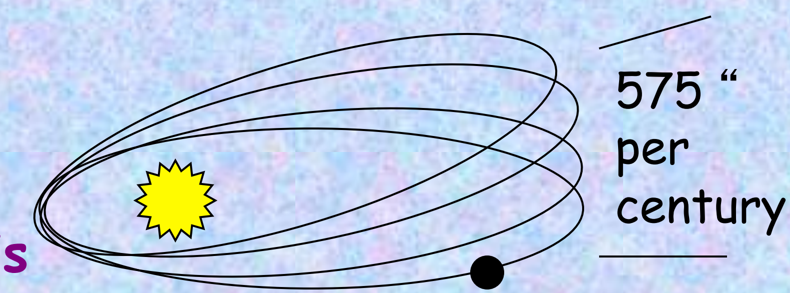
Planet	Semi-major axis (AU)	Orbital period (yr)	Eccentricity	Inclination to ecliptic $^\circ . '.''$	Inverse mass $1/M_\odot = 1$
Mercury	0.387099	0.24085	0.205628	7.0.15	6010000
Venus	0.723332	0.61521	0.006787	3.23.40	408400
Earth	1.000000	1.00004	0.016722	0.0.0	328910
Mars	1.523691	1.88089	0.093377	1.51.0	3098500
Jupiter	5.202803	11.86223	0.04845	1.18.17	1047.39
Saturn	9.53884	29.4577	0.05565	2.29.22	3498.5

For Jupiter:
$d\varpi/dt = 154 \,\text{as per century}\, (153.6)$

For Earth
$d\varpi/dt = 62 \,\text{as per century}\, (90)$

Mercury's Perihelion: Trouble to Triumph

- 1687 Newtonian triumph
- 1859 Leverrier's conundrum
- 1900 A turn-of-the century crisis



Planet	Advance
Venus	277.8
Earth	90.0
Mars	2.5
Jupiter	153.6
Saturn	7.3
Total	531.2

Perturbed Kepler problem

Worked example: perturbations by a third body

Case 2: the Kozai-Lidov mechanism

$$\langle \Delta a \rangle = 0$$

$$\langle \Delta e \rangle = \frac{15\pi}{2} \frac{m_3}{m} \left(\frac{a}{R} \right)^3 e(1-e^2)^{1/2} \sin^2 \iota \sin\omega \cos\omega$$

$$\langle \Delta \omega \rangle = \frac{3\pi}{2} \frac{m_3}{m} \left(\frac{a}{R} \right)^3 (1-e^2)^{-1/2} \left[5\cos^2 \iota \sin^2 \omega + (1-e^2)(5\cos^2\omega - 3) \right]$$

$$\langle \Delta \iota \rangle = -\frac{15\pi}{2} \frac{m_3}{m} \left(\frac{a}{R} \right)^3 e^2(1-e^2)^{-1/2} \sin\iota \cos\iota \sin\omega \cos\omega$$

Stationary point:

$$\omega_c = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$1 - e_c^2 = \frac{3}{5} \cos^2 \iota_c$$

A conserved quantity:

$$\frac{e}{1-e^2} \cos\iota \langle \Delta e \rangle + \sin\iota \langle \Delta \iota \rangle = 0$$

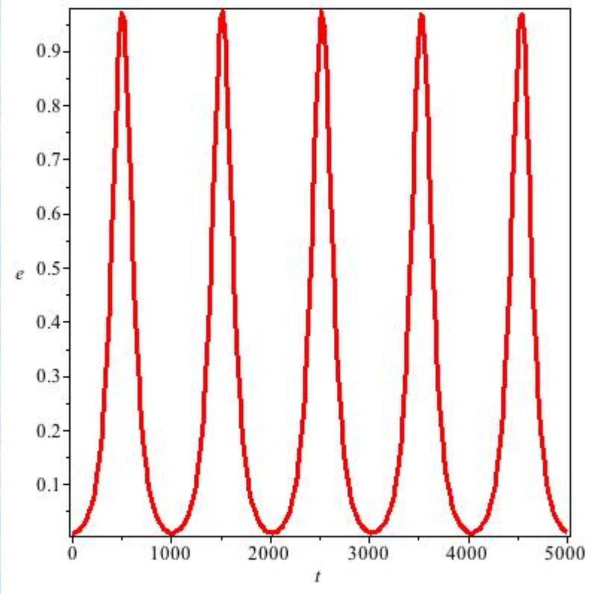
$$\Longrightarrow \sqrt{1-e^2} \cos\iota = \text{ constant}$$

L_z!

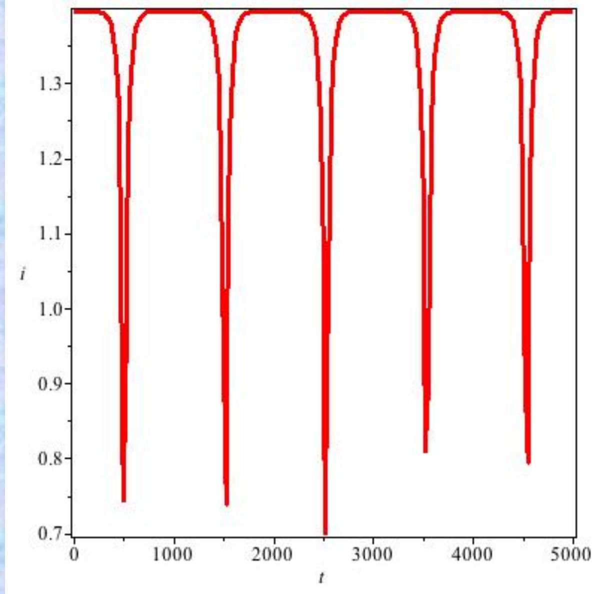
Perturbed Kepler problem

Worked example: perturbations by a third body

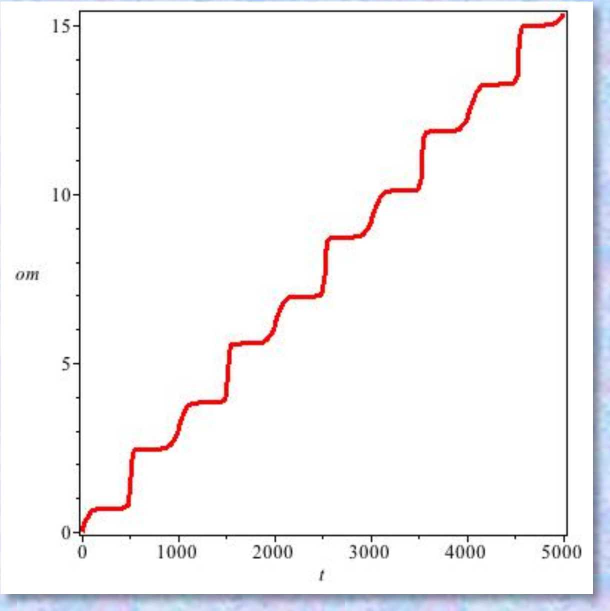
Case 2: the Kozai-Lidov mechanism



Eccentricity

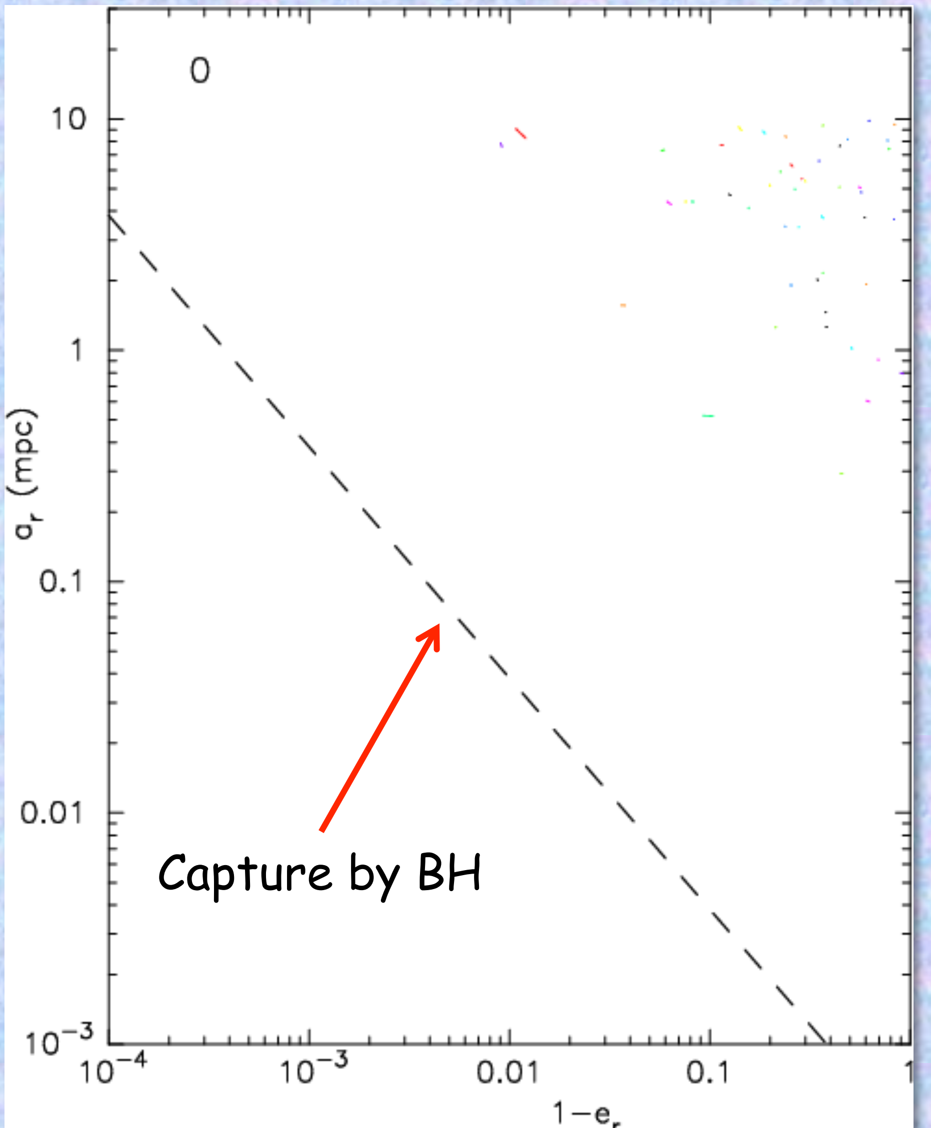


Inclination



Pericenter

Kozai oscillations and resonant relaxation



Merritt, Alexander, Mikkola & Will, PRD **84**, 044024 (2011)

Animations courtesy David Merritt

Perturbed Kepler problem

Worked example: body with a quadrupole moment

$$\boldsymbol{a} = -\frac{Gm\boldsymbol{r}}{r^3} - \frac{3}{2}J_2\frac{GmR^2}{r^4}\Big\{\big[5(\boldsymbol{e}\cdot\boldsymbol{n})^2 - 1\big]\boldsymbol{n} - 2(\boldsymbol{e}\cdot\boldsymbol{n})\boldsymbol{e}\Big\},$$

$$\Delta a = 0,\ \Delta e = 0,\ \Delta\iota = 0$$

$$\Delta\omega = 6\pi J_2\left(\frac{R}{p}\right)^2\left(1 - \frac{5}{4}\sin^2\iota\right)$$

$$\Delta\Omega = -3\pi J_2\left(\frac{R}{p}\right)^2\cos\iota$$

For Mercury (J_2 = 2.2 X 10^{-7})

$$\frac{d\varpi}{dt} = 0.03\,\mathrm{as/century}$$

For Earth satellites (J_2 = 1.08 X 10^{-3})

$$\frac{d\Omega}{dt} = -3639\cos\iota\left(\frac{R}{a}\right)^{7/2}\ \mathrm{deg/yr}$$

- LAGEOS (a=1.93 R, i = $109^{\circ}.8$): 120 deg/yr !
- Sun synchronous: a= 1.5 R, i = 65.9

"Curved spacetime tells matter how to move"

Continuous matter, stress energy tensor

Perfect fluid: $$T^{\alpha\beta} = (\rho c^2 + \epsilon + p)u^\alpha u^\beta / c^2 + p g^{\alpha\beta}$$

$$j^\alpha = \rho u^\alpha$$

$$\nabla_\beta T^{\alpha\beta} = 0\,,\ \nabla_\alpha j^\alpha = 0$$

ρ = rest mass density
ε = energy density
p = pressure
u^α = four velocity

1st law of Thermodynamics

$$u_\alpha \nabla_\beta T^{\alpha\beta} = 0 = \frac{d\varepsilon}{d\tau} + (\varepsilon + p)\nabla \cdot \vec{u}$$

$$d(\varepsilon \mathcal{V}) + p d\mathcal{V} = 0$$

Relativistic Euler equation

$$(\mu + p)\frac{Du^\alpha}{d\tau} = -c^2 \left(g^{\alpha\beta} + u^\alpha u^\beta / c^2\right) \nabla_\beta p$$

Compare with Newton

$$\rho \frac{d\boldsymbol{v}}{dt} - \boldsymbol{\nabla} U = -\boldsymbol{\nabla} p$$

FLORIDA GRAVITY

"Matter tells spacetime how to curve"

Riemann tensor $R^{\alpha}{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta} - \Gamma^{\alpha}_{\mu\delta}\Gamma^{\alpha}_{\beta\gamma}$

Ricci tensor $R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$

Ricci scalar $R = g^{\alpha\beta} R_{\alpha\beta}$

Einstein tensor $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$

Bianchi identities $\nabla_\beta G^{\alpha\beta} = 0$

Action $S = \frac{c^3}{16\pi G} \int \sqrt{-g} R d^4x + S_{\mathrm{matter}}$

Einstein's equations: $G^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta}$