Post-Newtonian limit of general relativity

$$g_{00} = -1 + \frac{2}{c^2}U + \frac{2}{c^4}\left(\psi + \frac{1}{2}\partial_{tt}X - U^2\right) + O(c^{-6})\,,$$

$$g_{0j} = -\frac{4}{c^3}U_j + O(c^{-5})\,,$$

$$g_{jk} = \delta_{jk}\left(1 + \frac{2}{c^2}U\right) + O(c^{-4})\,,$$

$$U(t,\boldsymbol{x}) := G\int \frac{\rho^{*\prime}}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x'\,,$$

$$\psi(t,\boldsymbol{x}) := G\int \frac{\rho^{*\prime}\left(\frac{3}{2}v'^2 - U' + \Pi' + 3p'/\rho^{*\prime}\right)}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x'\,,$$

$$X(t,\boldsymbol{x}) := G\int \rho^{*\prime}|\boldsymbol{x}-\boldsymbol{x'}|\,d^3x'\,,$$

$$U^j(t,\boldsymbol{x}) := G\int \frac{\rho^{*\prime}v'^j}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x'$$

Post-Newtonian Hydrodynamics

From $\nabla_\beta T^{\alpha\beta} = 0$

Post-Newtonian equation of hydrodynamics

$$\begin{aligned}
\rho^* \frac{dv^j}{dt} &= -\partial_j p + \rho^* \partial_j U \\
&\quad + \frac{1}{c^2}\left[\left(\frac{1}{2}v^2 + U + \Pi + \frac{p}{\rho^*}\right)\partial_j p - v^j \partial_t p\right] \\
&\quad + \frac{1}{c^2}\rho^*\Big[(v^2 - 4U)\partial_j U - v^j\left(3\partial_t U + 4v^k \partial_k U\right) \\
&\qquad + 4\partial_t U_j + 4v^k\left(\partial_k U_j - \partial_j U_k\right) + \partial_j \Psi\Big] \\
&\quad + O(c^{-4})
\end{aligned}$$

$$\Psi = \psi + \frac{1}{2}\partial_{tt} X$$

N-body equations of motion

Main assumptions:

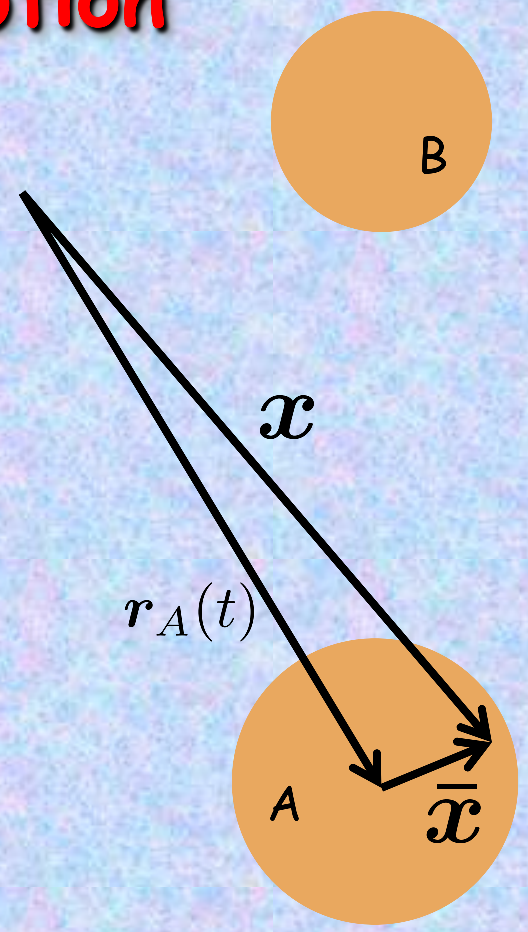
- Bodies small compared to typical separation (R << r)
- "isolated" -- no mass flow
- ignore contributions that scale as R^n
- assume bodies are reflection symmetric

$$\text{mass}: \quad m_A \equiv \int_A \rho^* d^3x$$

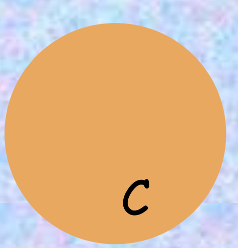
$$\text{position}: \quad \boldsymbol{r}_A(t) \equiv \frac{1}{m_A}\int_A \rho^* \boldsymbol{x} d^3x$$

$$\text{velocity}: \quad \boldsymbol{v}_A(t) \equiv \frac{1}{m_A}\int_A \rho^* \boldsymbol{v} d^3x = \frac{d\boldsymbol{r}_A}{dt}$$

$$\text{acceleration}: \quad \boldsymbol{a}_A(t) \equiv \frac{1}{m_A}\int_A \rho^* \boldsymbol{a} d^3x = \frac{d\boldsymbol{v}_A}{dt}$$



$$\boldsymbol{x} \equiv \boldsymbol{r}_A(t) + \bar{\boldsymbol{x}}$$



N-body equations of motion

$$\int_A \rho^* v^2 \partial_j U \, d^3x = \int_A \rho^* \left(v_A^2 + 2\boldsymbol{v}_A \cdot \bar{\boldsymbol{v}} + \bar{v}^2 \right) \partial_j U_A d^3x$$

$$+ \sum_B \int_A \rho^* \left(v_A^2 + 2\boldsymbol{v}_A \cdot \bar{\boldsymbol{v}} + \bar{v}^2 \right) \partial_j U_B d^3x$$

$$= 2H_A^{jk} v_A^k + \sum_B m_A v_A^2 \partial_j U_B + 2\mathcal{T}_A \sum_B \partial_j U_B$$

N-body equations of motion

Dependence on internal structure?

$$\mathcal{T}_A \equiv \frac{1}{2}\int_A \rho^* \bar{v}^2 \, d^3\bar{x}, \qquad P_A \equiv \int_A p \, d^3\bar{x}\,,$$

$$\Omega_A \equiv -\frac{1}{2}G\int_A \frac{\rho^* \rho^{*\prime}}{|\bar{\boldsymbol{x}} - \bar{\boldsymbol{x}}'|} \, d^3\bar{x}' d^3\bar{x}, \qquad E_A^{\rm int} \equiv \int_A \rho^* \Pi \, d^3\bar{x}$$

Use the virial theorem:

$$2\mathcal{T}_A + \Omega_A + 3P_A = 0$$

Then all structure integrals for body A disappear and those for the external bodies can be absorbed into a single "total" mass:

$$M_B \equiv m_B + \frac{1}{c^2}\left(\mathcal{T}_B + \Omega_B + E_B^{\rm int}\right) + O(c^{-4})$$

This is a manifestation of the **Strong Equivalence Principle,** satisfied by GR, but not by most alternative theories.
The motions of **all** bodies, including NS and BH, are independent of their internal structure – in GR!

N-body equations of motion

$$
\begin{aligned}
\boldsymbol{a}_A = &-\sum_{B\neq A}\frac{GM_B}{r_{AB}^2}\boldsymbol{n}_{AB}\\
&+\frac{1}{c^2}\Bigg(-\sum_{B\neq A}\frac{GM_B}{r_{AB}^2}\bigg[v_A^2-4(\boldsymbol{v}_A\cdot\boldsymbol{v}_B)+2v_B^2-\frac{3}{2}(\boldsymbol{n}_{AB}\cdot\boldsymbol{v}_B)^2\\
&\qquad\qquad\qquad\qquad -\frac{5GM_A}{r_{AB}}-\frac{4GM_B}{r_{AB}}\bigg]\boldsymbol{n}_{AB}\\
&+\sum_{B\neq A}\frac{GM_B}{r_{AB}^2}\Big[\boldsymbol{n}_{AB}\cdot(4\boldsymbol{v}_A-3\boldsymbol{v}_B)\Big](\boldsymbol{v}_A-\boldsymbol{v}_B)\\
&+\sum_{B\neq A}\sum_{C\neq A,B}\frac{G^2M_BM_C}{r_{AB}^2}\bigg[\frac{4}{r_{AC}}+\frac{1}{r_{BC}}-\frac{r_{AB}}{2r_{BC}^2}(\boldsymbol{n}_{AB}\cdot\boldsymbol{n}_{BC})\bigg]\boldsymbol{n}_{AB}\\
&-\frac{7}{2}\sum_{B\neq A}\sum_{C\neq A,B}\frac{G^2M_BM_C}{r_{AB}r_{BC}^2}\boldsymbol{n}_{BC}\Bigg)+O(c^{-4}).
\end{aligned}
$$

Post-Newtonian limit of general relativity

$$g_{00} = -1 + \frac{2}{c^2}U + \frac{2}{c^4}\left(\Psi - U^2\right) + O(c^{-6}),$$

$$g_{0j} = -\frac{4}{c^3}U_j + O(c^{-5}),$$

$$g_{jk} = \delta_{jk}\left(1 + \frac{2}{c^2}U\right) + O(c^{-4}),$$

$$\Psi = \psi + \frac{1}{2}\partial_{tt}X$$

$$U(t,\boldsymbol{x}) := G\int \frac{\rho^{*\prime}}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x',$$

$$\psi(t,\boldsymbol{x}) := G\int \frac{\rho^{*\prime}\left(\frac{3}{2}v'^2 - U' + \Pi' + 3p'/\rho^{*\prime}\right)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x',$$

$$X(t,\boldsymbol{x}) := G\int \rho^{*\prime}|\boldsymbol{x}-\boldsymbol{x}'|\,d^3x',$$

$$U^j(t,\boldsymbol{x}) := G\int \frac{\rho^{*\prime}v'^j}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x'$$

Post-Newtonian geodesic equation

$$\frac{d^2 r^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dr^\beta}{d\lambda}\frac{dr^\gamma}{d\lambda} = 0$$

λ = proper time (timelike)
= affine parameter (null)

Use 0 component $d^2t/d\lambda^2$ to change from λ to t

$$\frac{dv^\alpha}{dt} = -\left(\Gamma^\alpha_{\beta\gamma} - \frac{v^\alpha}{c}\Gamma^0_{\beta\gamma}\right)v^\beta v^\gamma$$

$v^\alpha = (c,\mathbf{v})$

Timelike test body $v^2 \sim U$

$$\frac{dv^j}{dt} = \partial_j U + \frac{1}{c^2}\Big[(v^2 - 4U)\partial_j U - \big(4v^k\partial_k U + 3\partial_t U\big)v^j - 4v^k\big(\partial_j U_k - \partial_k U_j\big) + 4\partial_t U_j + \partial_j \Psi\Big] + O(c^{-4})$$

Post-Newtonian geodesic equation

Lightlike test body $v^2 \sim c^2$

$$ds^2 = 0 = -\left(1 - \frac{2}{c^2}U + O(c^{-4})\right)c^2dt^2 + \left(1 + \frac{2}{c^2}U + O(c^{-4})\right)v^2dt^2 - \frac{8}{c^2}U^jv_jdt^2$$

$$v^2 = c^2\left(1 - \frac{4}{c^2}U + O(c^{-3})\right) \Rightarrow \boldsymbol{v} = c\left(1 - \frac{2}{c^2}U\right)\boldsymbol{n}$$

Geodesic equation with v ~ c

$$\frac{dv^j}{dt} = \left(1 + \frac{v^2}{c^2}\right)\partial_j U - \frac{4}{c^2}v^jv^k\partial_k U + O(c^{-3})$$

$$\frac{dn^j}{dt} = \frac{2}{c}(\delta^{jk} - n^jn^k)\partial_k U + O(c^{-2})$$

These will be used for the deflection of light, Shapiro time delay and gravitational lenses

N-body equations of motion

$$
\begin{aligned}
\boldsymbol{a}_A = &-\sum_{B\neq A}\frac{GM_B}{r_{AB}^2}\boldsymbol{n}_{AB} \\
&+\frac{1}{c^2}\Biggl(-\sum_{B\neq A}\frac{GM_B}{r_{AB}^2}\biggl[v_A^2-4(\boldsymbol{v}_A\cdot\boldsymbol{v}_B)+2v_B^2-\frac{3}{2}(\boldsymbol{n}_{AB}\cdot\boldsymbol{v}_B)^2 \\
&\qquad\qquad -\frac{5GM_A}{r_{AB}}-\frac{4GM_B}{r_{AB}}\biggr]\boldsymbol{n}_{AB} \\
&+\sum_{B\neq A}\frac{GM_B}{r_{AB}^2}\Bigl[\boldsymbol{n}_{AB}\cdot(4\boldsymbol{v}_A-3\boldsymbol{v}_B)\Bigr](\boldsymbol{v}_A-\boldsymbol{v}_B) \\
&+\sum_{B\neq A}\sum_{C\neq A,B}\frac{G^2M_BM_C}{r_{AB}^2}\biggl[\frac{4}{r_{AC}}+\frac{1}{r_{BC}}-\frac{r_{AB}}{2r_{BC}^2}(\boldsymbol{n}_{AB}\cdot\boldsymbol{n}_{BC})\biggr]\boldsymbol{n}_{AB} \\
&-\frac{7}{2}\sum_{B\neq A}\sum_{C\neq A,B}\frac{G^2M_BM_C}{r_{AB}r_{BC}^2}\boldsymbol{n}_{BC}\Biggr)+O(c^{-4}).
\end{aligned}
$$

PN 2-body equations of motion

Define:

$$\boldsymbol{r} \equiv \boldsymbol{r}_1 - \boldsymbol{r}_2 \qquad m \equiv M_1 + M_2$$

$$\boldsymbol{v} \equiv \boldsymbol{v}_1 - \boldsymbol{v}_2 \qquad \eta \equiv \frac{M_1 M_2}{(M_1+M_2)^2}$$

$$\boldsymbol{a} \equiv \boldsymbol{a}_1 - \boldsymbol{a}_2$$

$$\boldsymbol{n} \equiv \boldsymbol{r}/r$$

$$\dot{r} \equiv dr/dt = \boldsymbol{n}\cdot\boldsymbol{v}$$

$$\boldsymbol{a} = -\frac{Gm}{r^2}\boldsymbol{n} - \frac{Gm}{c^2 r^2}\left\{ \left[(1+3\eta)v^2 - \frac{3}{2}\eta\dot{r}^2 - 2(2+\eta)\frac{Gm}{r} \right]\boldsymbol{n} - 2(2-\eta)\dot{r}\boldsymbol{v} \right\} + O(c^{-4}),$$

- Geodesic equation: $\eta = 0$, more general potentials U, ψ, X
- PN 2-body equation: general masses, ignore moments

PN 2-body equations of motion: conserved quantities

Conserved energy: dot v into the equation of motion

$$\frac{1}{2}\frac{d}{dt}v^2 = -\frac{Gm}{r^2}\dot{r} + \frac{Gm}{c^2r^2}\dot{r}\left[(3-5\eta)v^2 + \frac{3}{2}\eta\dot{r}^2 + 2(2+\eta)\frac{Gm}{r}\right]$$

$$= \frac{d}{dt}\left(\frac{Gm}{r}\right) - \frac{d}{dt}\left\{\frac{Gm}{c^2r}\left[(3-4\eta)v^2 - \left(1-\frac{9}{2}\eta\right)\frac{Gm}{r} + \frac{1}{2}\eta\dot{r}^2\right]\right\}$$

$$\frac{(Gm)^2}{r^3}\dot{r} = -\frac{1}{2}\frac{d}{dt}\left(\frac{Gm}{r}\right)^2$$

$$\frac{Gm}{r^2}\dot{r}v^2 = \frac{d}{dt}\left[\frac{Gm}{r}\left(\frac{Gm}{r} - v^2\right)\right]$$

$$\frac{Gm}{r^2}\dot{r}^3 = \frac{1}{3}\frac{d}{dt}\left[\frac{Gm}{r}\left(3\frac{Gm}{r} - 2v^2 - \dot{r}^2\right)\right]$$

PN 2-body equations of motion: conserved quantities

Conserved energy: dot v into the equation of motion

$$\frac{1}{2}\frac{d}{dt}v^2 = -\frac{Gm}{r^2}\dot{r} + \frac{Gm}{c^2r^2}\dot{r}\left[(3-5\eta)v^2 + \frac{3}{2}\eta\dot{r}^2 + 2(2+\eta)\frac{Gm}{r}\right]$$

$$= \frac{d}{dt}\left(\frac{Gm}{r}\right) - \frac{d}{dt}\left\{\frac{Gm}{c^2r}\left[(3-4\eta)v^2 - \left(1-\frac{9}{2}\eta\right)\frac{Gm}{r} + \frac{1}{2}\eta\dot{r}^2\right]\right\}$$

$$-\frac{3}{2c^2}(1-3\eta)\frac{d}{dt}\left(\frac{1}{2}v^2 - \frac{Gm}{r}\right)^2$$

$$\varepsilon := \frac{1}{2}v^2 - \frac{Gm}{r} + \frac{1}{c^2}\left\{\frac{3}{8}(1-3\eta)v^4 + \frac{Gm}{2r}\left[(3+\eta)v^2 + \eta\dot{r}^2 + \frac{Gm}{r}\right]\right\} + O(c^{-4})$$

$$E_{\rm SR} = \gamma_1 m_1 c^2 + \gamma_2 m_2 c^2$$

PN 2-body equations of motion: conserved quantities

Conserved angular momentum: cross x with the equation of motion

$$\frac{d}{dt}(\boldsymbol{r}\times\boldsymbol{v}) = (4-2\eta)\frac{Gm}{c^2r^2}\dot{r}(\boldsymbol{r}\times\boldsymbol{v})$$

$$= -\frac{d}{dt}\left[(4-2\eta)\frac{Gm}{c^2r}(\boldsymbol{r}\times\boldsymbol{v})\right]$$

$$-\frac{d}{dt}\left[\frac{1-3\eta}{c^2}\left(\frac{1}{2}v^2-\frac{Gm}{c^2r}\right)(\boldsymbol{r}\times\boldsymbol{v})\right]$$

$$\boldsymbol{h} = \left\{1+\frac{1}{c^2}\left[\frac{1}{2}(1-3\eta)v^2+(3+\eta)\frac{Gm}{r}\right]\right\}(\boldsymbol{r}\times\boldsymbol{v})+O(c^{-4})$$

$$\boldsymbol{L}_{\rm SR} = \gamma_1 m_1(\boldsymbol{x}_1\times\boldsymbol{v}_1)+\gamma_2 m_2(\boldsymbol{x}_2\times\boldsymbol{v}_2)$$

FLORIDA GRAVITY

PN circular orbits

$$\boldsymbol{a} = -\frac{Gm}{r^2}\boldsymbol{n} - \frac{Gm}{c^2 r^2}\left\{\left[(1+3\eta)v^2 - \frac{3}{2}\eta\dot{r}^2 - 2(2+\eta)\frac{Gm}{r}\right]\boldsymbol{n} - 2(2-\eta)\dot{r}\boldsymbol{v}\right\} + O(c^{-4}),$$

$$d\boldsymbol{n}/dt = \dot{\phi}\boldsymbol{\lambda}$$
$$d\boldsymbol{\lambda}/dt = -\dot{\phi}\boldsymbol{n}$$
$$\boldsymbol{v} = \dot{r}\boldsymbol{n} + r\dot{\phi}\boldsymbol{\lambda}$$
$$\boldsymbol{a} = (\ddot{r} - r\dot{\phi}^2)\boldsymbol{n} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\boldsymbol{\lambda}$$

$$\dot{r} = 0 \Longrightarrow \begin{cases} \ddot{r} = 0 \\ \dot{\phi} = \Omega = \text{const} \end{cases}$$

$$\Omega^2 = \frac{Gm}{r^3}\left[1 - (3-\eta)\frac{Gm}{c^2 r} + O(c^{-4})\right]$$

$$\varepsilon = -\frac{Gm}{2r}\left[1 - \frac{1}{4}(7-\eta)\frac{Gm}{c^2 r} + O(c^{-4})\right]$$

$$h = \sqrt{Gmr}\left[1 + 2\frac{Gm}{c^2 r} + O(c^{-4})\right]$$

Remark: test body in Schwarzschild coords ($\eta = 0$, $r = r_S - Gm/c^2$)

$$\Omega^2 = \frac{Gm}{r_S^3}$$

N-body equations of motion:
Worked example: 2 bodies and the perihelion shift

Components of the disturbing force

$$\mathcal{R} = \frac{Gm}{c^2 r^2}\left[-(1+3\eta)v^2 + \frac{1}{2}(8-\eta)\dot{r}^2 + 2(2+\eta)\frac{Gm}{r}\right],$$

$$\mathcal{S} = \frac{Gm}{c^2 r^2}\left[2(2-\eta)\dot{r}(r\dot{\phi})\right],$$

$$\mathcal{W} = 0$$

Integrate the Lagrange planetary equations:

$$\Delta e = \Delta a = 0$$

$$\Delta\Omega = \Delta\iota = 0$$

$$\Delta\omega = \frac{6\pi G(M_1 + M_2)}{a(1-e^2)c^2}$$

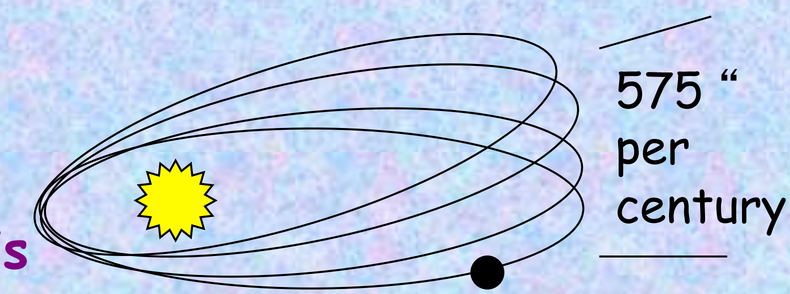
42.98 "/c for Mercury

4.226598 °/yr for PSR 1913+16

FLORIDA GRAVITY

Mercury's Perihelion: Trouble to Triumph

- 1687 Newtonian triumph
- 1859 Leverrier's conundrum
- 1900 A turn-of-the century crisis



Planet	Advance
Venus	277.8
Earth	90.0
Mars	2.5
Jupiter	153.6
Saturn	7.3
Total	531.2
Discrepancy	42.9
Modern measured value	42.982 ± 0.001
General relativity prediction	42.98

Deflection of light

$$\boldsymbol{v} = c\left(1 - \frac{2}{c^2}U\right)\boldsymbol{n} \qquad \frac{dn^j}{dt} = \frac{2}{c}(\delta^{jk} - n^j n^k)\partial_k U \qquad U = G\int \frac{\rho'}{|\boldsymbol{r}(t) - \boldsymbol{x}'|}d^3x'$$

Zeroth order trajectory:

$$\boldsymbol{r}(t) = \boldsymbol{r}_e + c\boldsymbol{k}(t - t_e) + O(c^{-2})$$

To PN order, let

$$\boldsymbol{n} = \boldsymbol{k} + \boldsymbol{\alpha} + O(c^{-4})$$

$$\boldsymbol{s}_e = \boldsymbol{r}_e - \boldsymbol{x}'$$

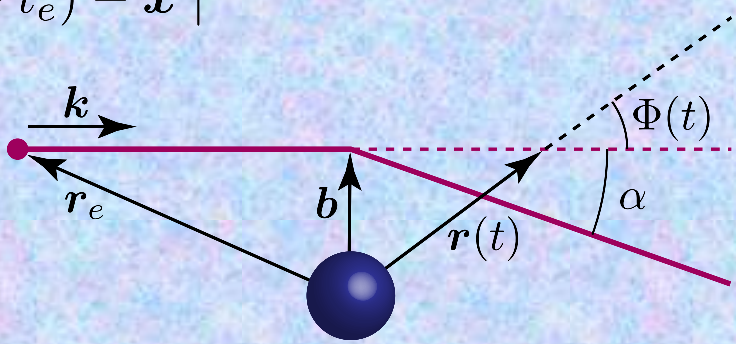
$$\boldsymbol{s} = \boldsymbol{r}(t) - \boldsymbol{x}'$$

$$\boldsymbol{b} = \boldsymbol{s}_e - \boldsymbol{k}(\boldsymbol{s}_e \cdot \boldsymbol{k})$$

$$\frac{d\alpha^j}{dt} = -\frac{2G}{c}(\delta^{jk} - k^j k^k)\int \rho' \frac{r_e^k + ck^k(t - t_e) - x'}{|\boldsymbol{r}_e + c\boldsymbol{k}(t - t_e) - \boldsymbol{x}'|^3}d^3x'$$

$$= -\frac{2G}{c}\int \rho' \frac{\boldsymbol{b}}{s^3}d^3x'$$

$$= -\frac{2G}{c^2}\int \rho' \frac{\boldsymbol{b}}{b^2}\frac{d}{dt}\left(\frac{\boldsymbol{s}\cdot\boldsymbol{k}}{s}\right)d^3x'$$



$$\boldsymbol{\alpha}(t) = -\frac{2G}{c^2}\int \rho(t, \boldsymbol{x}')\frac{\boldsymbol{b}}{b^2}\left(\frac{\boldsymbol{s}\cdot\boldsymbol{k}}{s} - \frac{\boldsymbol{s}_e\cdot\boldsymbol{k}}{s_e}\right)d^3x'$$

FLORIDA GRAVITY

Deflection of light

For a point mass at the origin

$$\boldsymbol{\alpha}(t) = -\frac{2GM}{c^2}\frac{\boldsymbol{b}}{b^2}\left[\frac{\boldsymbol{r}(t)\cdot\boldsymbol{k}}{r(t)} - \frac{\boldsymbol{r}_e\cdot\boldsymbol{k}}{r_e}\right]$$

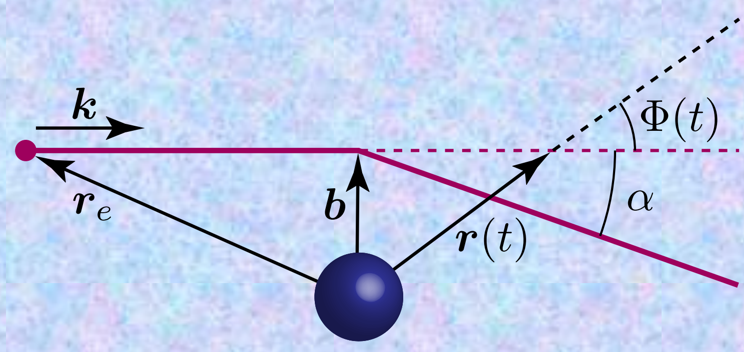
$$\approx -\frac{4GM}{c^2 b}\hat{\boldsymbol{b}}\left[\frac{\cos\Phi(t)+1}{2}\right]$$

For $\Phi \approx 0$

$$\Delta\theta \approx |\boldsymbol{\alpha}|$$

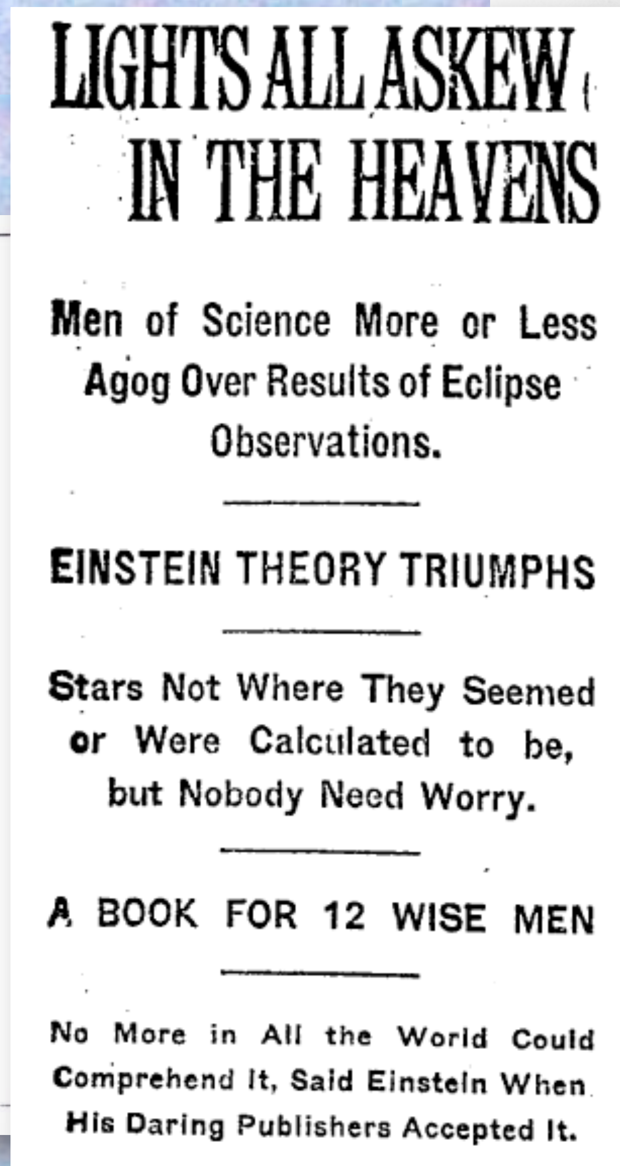
$$= \frac{4GM}{c^2 b}$$

$$= 1''.7505\left(\frac{M/M_\odot}{b/R_\odot}\right)$$



Deflection of light: The 1919 Eclipse

A. S. Eddington



LIGHTS ALL ASKEW IN THE HEAVENS

Men of Science More or Less Agog Over Results of Eclipse Observations.

EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

A BOOK FOR 12 WISE MEN

No More in All the World Could Comprehend It, Said Einstein When His Daring Publishers Accepted It.

Sobral site

Photo from Principe

Deflection of light: Results