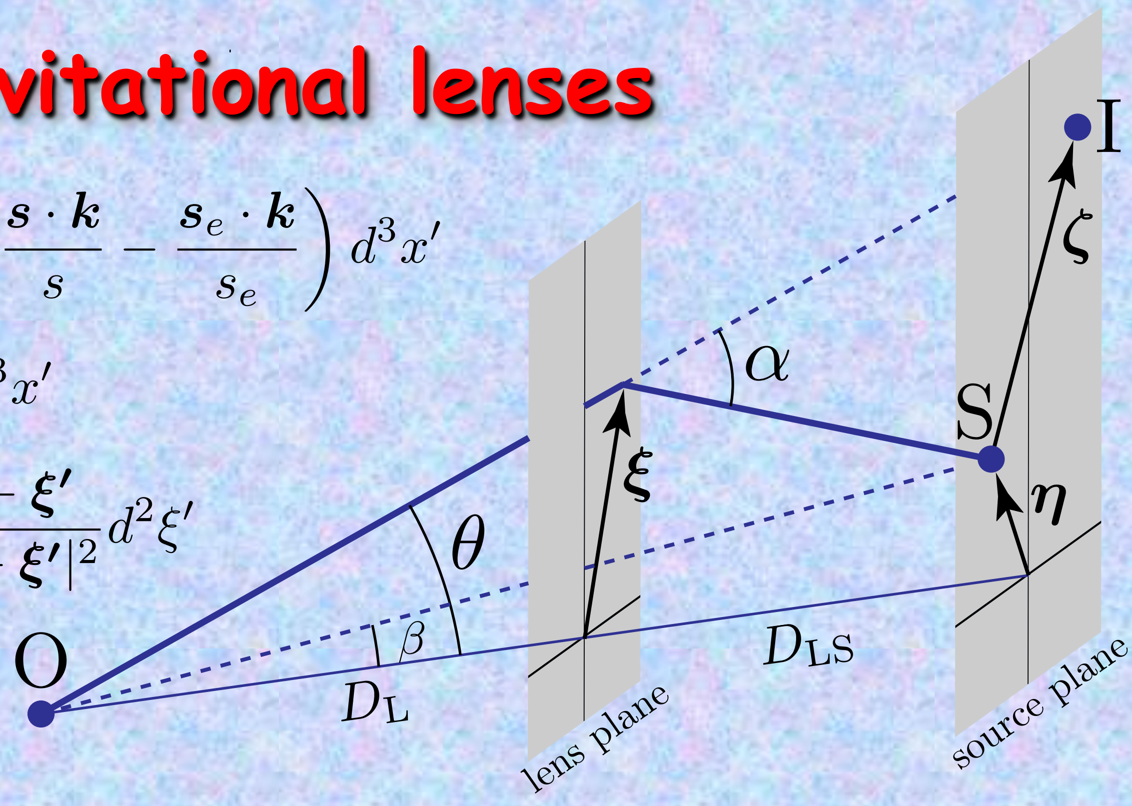
Gravitational lenses

$$\boldsymbol{\alpha}(t) = -\frac{2G}{c^2}\int \rho(t,\boldsymbol{x'})\frac{\boldsymbol{b}}{b^2}\left(\frac{\boldsymbol{s}\cdot\boldsymbol{k}}{s} - \frac{\boldsymbol{s}_e\cdot\boldsymbol{k}}{s_e}\right) d^3x'$$

$$= -\frac{4G}{c^2}\int \rho(t,\boldsymbol{x'})\frac{\boldsymbol{b}}{b^2} d^3x'$$

$$= -\frac{4G}{c^2}\int \Sigma(t,\boldsymbol{\xi'})\frac{\boldsymbol{\xi}-\boldsymbol{\xi'}}{|\boldsymbol{\xi}-\boldsymbol{\xi'}|^2} d^2\xi'$$



$$D_{\rm S}\boldsymbol{\theta} = \boldsymbol{\eta} + \boldsymbol{\zeta} = D_{\rm S}\boldsymbol{\beta} - D_{\rm LS}\boldsymbol{\alpha}$$

Lens equation

$$\boldsymbol{\theta} + \frac{D_{\rm LS}}{D_{\rm S}}\boldsymbol{\alpha} = \boldsymbol{\beta}$$

$$\boldsymbol{\xi} = D_{\rm L}\boldsymbol{\theta}$$

Schwarzschild lens

$$\boldsymbol{\alpha}(\boldsymbol{\xi}) = -\frac{4GM}{c^2}\frac{\boldsymbol{\xi}}{\xi^2}$$

$$\theta - \frac{\theta_{\rm E}^2}{\theta} = \beta$$

$$\theta_{\rm E}^2 := \frac{4GM}{c^2}\frac{D_{\rm LS}}{D_{\rm S}D_{\rm L}}$$

Gravitational lenses

$$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_\mathrm{E}^2}\right)$$

$$\theta_\mathrm{E} \sim 2\,\mathrm{as} \qquad M \sim 10^{12} M_\odot$$

$$\sim 0.5\,\mathrm{mas} \qquad M \sim M_\odot$$

$$\beta \ll \theta_\mathrm{E}, \quad \theta_\pm = \pm\theta_\mathrm{E} + \beta/2$$

$$\beta = 0, \quad \theta_\pm = \pm\theta_\mathrm{E} \quad \text{Einstein ring}$$

$$\beta \gg \theta_\mathrm{E}, \quad \theta_\pm = \begin{cases} \beta + \theta_\mathrm{E}^2/\beta \\ -\theta_\mathrm{E}^2/\beta \end{cases}$$

Gravitational lenses: Magnification

$$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_{\rm E}^2}\right)$$

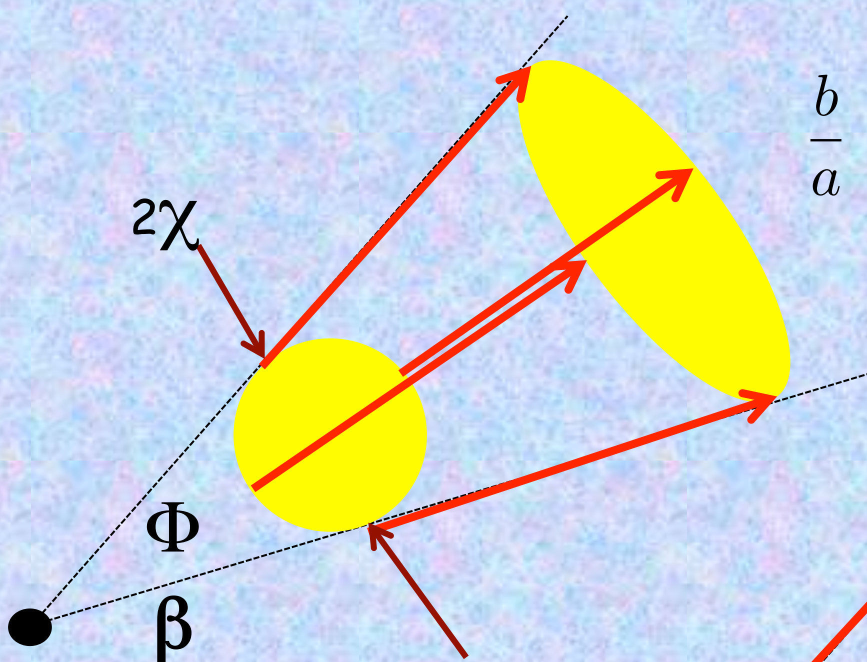
Brightness proportional to $\int \beta d\beta d\varphi$ at the source, $\int \theta d\theta d\varphi$ at the observer

$$\mu_\pm = \frac{\theta_\pm d\theta_\pm}{\beta d\beta} = \pm\frac{1}{4}\left(\frac{\beta}{\sqrt{\beta^2 + 4\theta_{\rm E}^2}} + \frac{\sqrt{\beta^2 + 4\theta_{\rm E}^2}}{\beta} \pm 2\right).$$

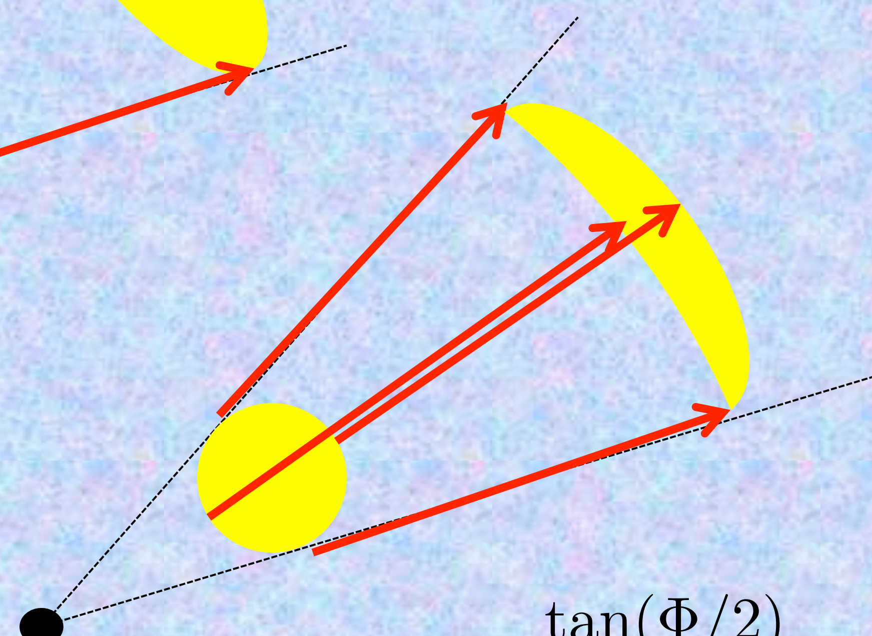
Microlensing

$$|\mu_+| + |\mu_-| = \frac{1}{2}\left(\frac{\beta}{\sqrt{\beta^2 + 4\theta_{\rm E}^2}} + \frac{\sqrt{\beta^2 + 4\theta_{\rm E}^2}}{\beta}\right) > 1$$

Gravitational lenses: Distortion

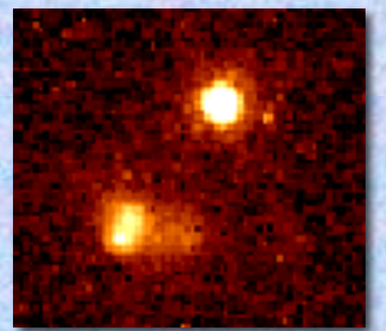
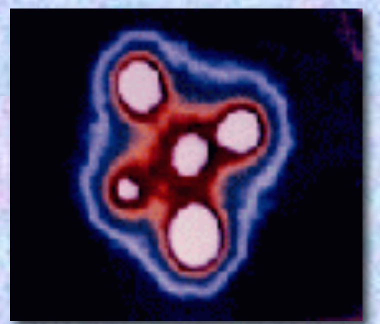


$$\frac{b}{a} = \frac{\beta}{\sqrt{\beta^2 + 4\theta_{\rm E}^2}} \quad (\chi \ll \beta)$$

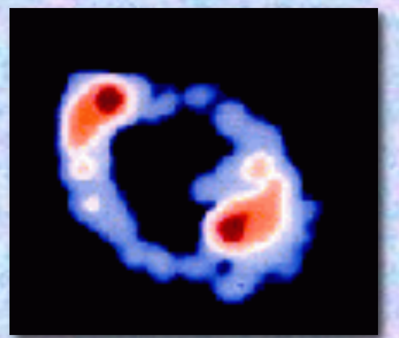


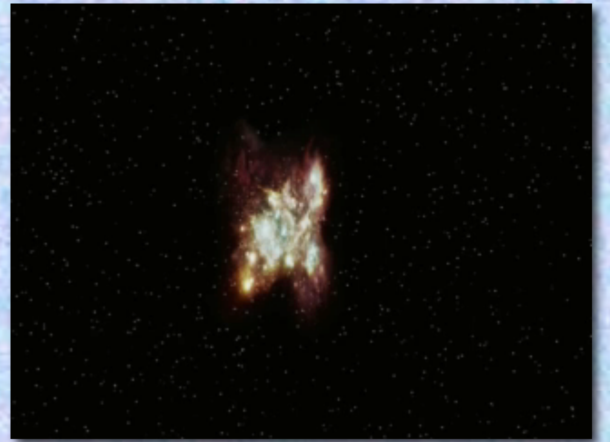
$$\beta_c = 2\Theta_{\rm E}\frac{\tan(\Phi/2)}{1 - \sin(\Phi/2)}$$

Gravitational lenses





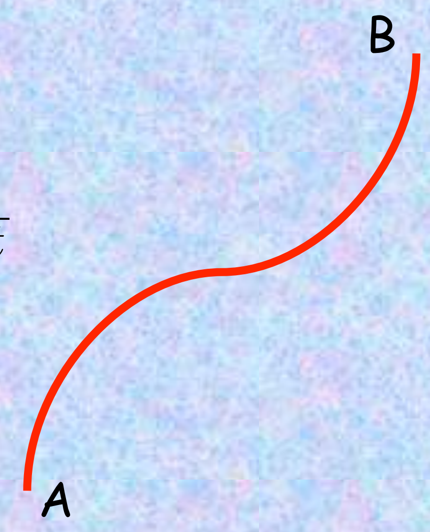
Post-Newtonian time

Clock synchronization on the Earth

$$d\tau = \left[1 - \frac{1}{c^2}\left(\frac{1}{2}\bar{v}^2 + U\right) + O(c^{-4})\right] d\bar{t}$$

$$= \left[1 - \frac{1}{c^2}\left(\frac{1}{2}v^2 + \boldsymbol{v}\cdot(\boldsymbol{\omega}\times\boldsymbol{r}) + \Phi\right) + O(c^{-4})\right] d\bar{t}$$

$$\Phi = U + \frac{1}{2}\omega^2(x^2+y^2)$$ Geoid potential



Two clocks at rest

$$\tau_A = \left(1 - \frac{\Phi_{\rm geoid}}{c^2}\right)\bar{t}$$

$$\tau_B = \left(1 - \frac{\Phi_{\rm geoid}}{c^2}\right)\bar{t} + \tau_B^0$$

For travelling clock with v << c

$$\tau_{\rm trav} = \tau_A(\bar{t}_1) + \left(1 - \frac{\Phi_{\rm geoid}}{c^2}\right)(\bar{t}_2 - \bar{t}_1) - \frac{1}{c^2}\int_A^B (\boldsymbol{\omega}\times\boldsymbol{r})\cdot d\boldsymbol{r}$$

$$\tau_B^0 = -\frac{1}{c^2}\int_A^B (\boldsymbol{\omega}\times\boldsymbol{r})\cdot d\boldsymbol{r}$$

Sagnac effect:
207 ns for closed path
at the equator

Post-Newtonian time: GPS

$$d\tau_{\rm ground} = \left(1 - \frac{\Phi_{\rm geoid}}{c^2}\right) d\bar{t}$$

$$d\tau_{\rm sat} = \left[1 - \frac{1}{c^2}\left(\frac{1}{2}v^2 + U\right)\right] d\bar{t}$$

$$= \left(1 - \frac{3}{2}\frac{Gm}{c^2 a}\right) d\bar{t}$$

$$d\tau_{\rm sat} \left(1 + \frac{\Phi_{\rm geoid}}{c^2} - \frac{3}{2}\frac{Gm}{c^2 a}\right) d\tau_{\rm ground}$$

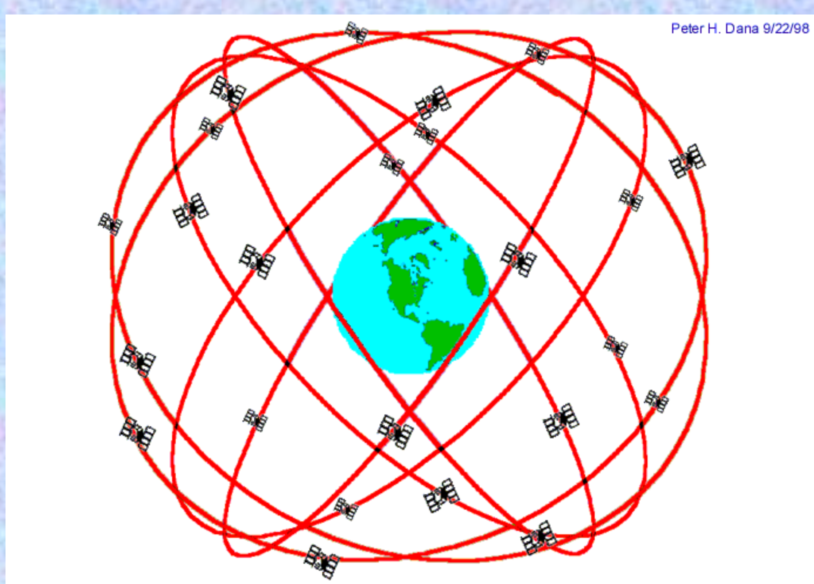
$$\sim \frac{Gm}{c^2 a_\oplus} \sim 6.9 \times 10^{-10}$$

$$\sim \frac{3}{2}\frac{Gm}{c^2(4.2 a_\oplus)} \sim 2.5 \times 10^{-10}$$

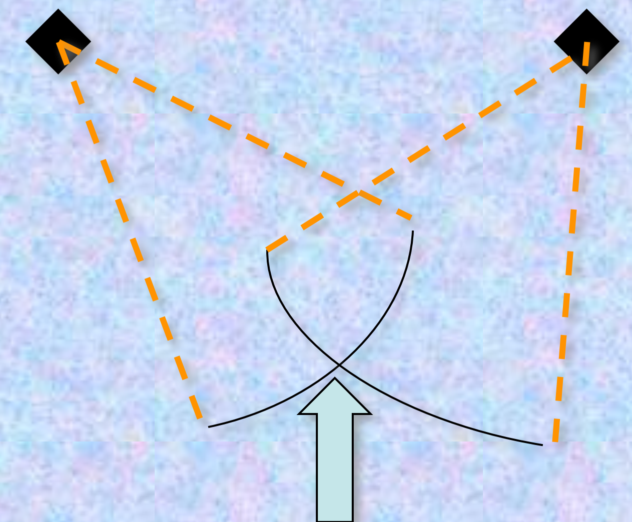
for GPS satellites

General Relativity and Daily Life

The Global Positioning System (GPS)



GPS Nominal Constellation
24 Satellites in 6 Orbital Planes
4 Satellites in each Plane
20,200 km Altitudes, 55 Degree Inclination



Navigation Requirement: 15 m ➪ 50ns

Relativity **must** be taken into account, for GPS to function

Relativistic effects: 39,000 ns per day!

GR = 46,000

SR = -7,000