PN theory: Far-zone physics

Recall the wave-zone solution to $\Box\psi = -4\pi\mu$

$$\frac{\mu(t-|\boldsymbol{x}-\boldsymbol{x'}|/c,\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{x'}|} = \sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}x'^L\partial_L\frac{\mu(t-r/c,\boldsymbol{y})}{r}.$$

$$\psi_{\mathcal{N}}(t,\boldsymbol{x}) = \sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\int_{\mathcal{M}}\mu(\tau,\boldsymbol{x'})x'^L\,d^3x'\right]$$

Must add the integral over the wave zone $\psi_{\mathcal{W}}$

For the field $h^{\alpha\beta}$:

$$h^{\alpha\beta}_{\mathcal{N}}(t,\boldsymbol{x}) = \frac{4G}{c^4}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\int_{\mathcal{M}}\tau^{\alpha\beta}(\tau,\boldsymbol{x'})x'^L\,d^3x'\right]$$

General form of wave-zone fields

M_0 $\quad$ $M_0 X^k = 0$

$$h^{00}_{\mathcal{N}}(t,\boldsymbol{x}) = \frac{4G}{c^4}\left\{\frac{1}{r}\int_{\mathcal{M}} \tau^{00} d^3x' - \partial_k\left[\frac{1}{r}\int_{\mathcal{M}} \tau^{00}(\tau,\boldsymbol{x'}) x^k \, d^3x'\right] + \sum_{\ell=2}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\int_{\mathcal{M}} \tau^{00}(\tau,\boldsymbol{x'}) x'^L \, d^3x'\right]\right\}$$

$\mathcal{I}^L(\tau)$

$$h^{00}_{\mathcal{N}} = \frac{4GM_0}{c^2 r} + \frac{4G}{c^2}\sum_{\ell=2}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left[\frac{\mathcal{I}^L(\tau)}{r}\right]$$

General form of wave-zone fields

$$\tau^{0j}x^k = \frac{1}{2}\partial_0\left(\tau^{00}x^jx^k\right) + \tau^{0[j}x^{k]} + \frac{1}{2}\partial_p\left(\tau^{0p}x^jx^k\right)$$

$P^j = 0$

$$h^{0j}_{\mathcal{N}}(t,\boldsymbol{x}) = \frac{4G}{c^4}\left\{\frac{1}{r}\int_{\mathcal{M}}\tau^{0j}\,d^3x' - \partial_k\left[\frac{1}{r}\int_{\mathcal{M}}\tau^{0j}(\tau,\boldsymbol{x'})x^k\,d^3x'\right] + \sum_{\ell=2}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\int_{\mathcal{M}}\tau^{0j}(\tau,\boldsymbol{x'})x'^L\,d^3x'\right]\right\}$$

$$h^{0j}_{\mathcal{N}} = -\frac{2G}{c^3}\frac{(\boldsymbol{n}\times\boldsymbol{J}_0)^j}{r^2} - \frac{2G}{c^3}\partial_k\left[\frac{\dot{\mathcal{I}}^{jk}(\tau)}{r}\right] + \frac{4G}{c^4}\sum_{\ell=2}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\int_{\mathcal{M}}\tau^{0j}(\tau,\boldsymbol{x'})x'^L\,d^3x'\right]$$

General form of wave-zone fields

$$h^{jk}_{\mathcal{N}}(t,\boldsymbol{x}) = \frac{4G}{c^4}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\int_{\mathcal{M}}\tau^{jk}(\tau,\boldsymbol{x'})x'^L\,d^3x'\right]$$

$$\tau^{jk} = \frac{1}{2}\partial_{00}\left(\tau^{00}x^j x^k\right) + \frac{1}{2}\partial_p\left(2\tau^{p(j}x^{k)} - \partial_q\tau^{pq}x^j x^k\right)$$

$$h^{jk}_{\mathcal{N}} = \frac{2G}{c^4}\frac{\ddot{\mathcal{I}}^{jk}(\tau)}{r} + \frac{4G}{c^4}\sum_{\ell=1}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\int_{\mathcal{M}}\tau^{jk}(\tau,\boldsymbol{x'})x'^L\,d^3x'\right]$$

General form of wave-zone fields

In the far-away wave-zone, keep only 1/r terms

$$\partial_k r = n_k$$
$$\partial_k r^{-1} = -\frac{n_k}{r^2}$$
$$\partial_j n_k = \frac{1}{r}(\delta_{jk} - n_j n_k)$$
$$\partial_k \tau = -\frac{1}{c} n_k$$

$$h^{00}_{\mathcal{N}} = \frac{4GM}{c^2 r} + \frac{4G}{c^2}\sum_{\ell=2}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left[\frac{\mathcal{I}^L(\tau)}{r}\right]$$

$$\Longrightarrow \frac{4GM}{c^2 r} + \frac{4G}{rc^2}\sum_{\ell=2}^{\infty}\frac{n^L}{c^\ell \ell!}\frac{d^\ell}{d\tau^\ell}\mathcal{I}^L(\tau)$$

$$h^{0j}_{\mathcal{N}} = -\frac{2G}{c^3}\frac{(\boldsymbol{n}\times\boldsymbol{J}_0)^j}{r^2} - \frac{2G}{c^3}\partial_k\left[\frac{\dot{\mathcal{I}}^{jk}(\tau)}{r}\right] + \frac{4G}{c^4}\sum_{\ell=2}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\int_{\mathcal{M}}\tau^{0j}(\tau,\boldsymbol{x}')x'^L\,d^3x'\right]$$

$$\Longrightarrow \frac{2G}{rc^4}n_k\ddot{\mathcal{I}}^{jk}(\tau) + \frac{4G}{rc^4}\sum_{\ell=2}^{\infty}\frac{n^L}{c^\ell \ell!}\frac{d^\ell}{d\tau^\ell}\mathcal{M}^{0jL}(\tau)$$

$$h^{jk}_{\mathcal{N}} = \frac{2G}{c^4}\frac{\ddot{\mathcal{I}}^{jk}(\tau)}{r} + \frac{4G}{c^4}\sum_{\ell=1}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\int_{\mathcal{M}}\tau^{jk}(\tau,\boldsymbol{x}')x'^L\,d^3x'\right]$$

$$\Longrightarrow \frac{2G}{rc^4}\ddot{\mathcal{I}}^{jk}(\tau) + \frac{4G}{rc^4}\sum_{\ell=1}^{\infty}\frac{n^L}{c^\ell \ell!}\frac{d^\ell}{d\tau^\ell}\mathcal{M}^{jkL}(\tau)$$

$$h^{00} = \frac{4GM}{c^2 R} + \frac{G}{c^4 R}C(\tau,\boldsymbol{N})$$
$$h^{0j} = \frac{G}{c^4 R}D^j(\tau,\boldsymbol{N})$$
$$h^{jk} = \frac{G}{c^4 R}A^{jk}(\tau,\boldsymbol{N})$$

Far wave-zone fields & TT gauge

$$\partial_j h^{\alpha\beta} = -\frac{1}{c} N_j \partial_\tau h^{\alpha\beta} + O(R^{-2})$$

$$h^{00} = \frac{4GM}{c^2 R} + \frac{G}{c^4 R} C(\tau, \boldsymbol{N})$$
$$h^{0j} = \frac{G}{c^4 R} D^j(\tau, \boldsymbol{N})$$
$$h^{jk} = \frac{G}{c^4 R} A^{jk}(\tau, \boldsymbol{N})$$

(from now on, r -> R, n^j -> N^j)

Harmonic gauge

$$\partial_\beta h^{\alpha\beta} = 0 = \frac{1}{c} \left(\partial_\tau h^{\alpha 0} - N_j \partial_\tau h^{\alpha j} \right) + O(R^{-2})$$

Integrating:

$$C = N_j D^j = N_j N_k A^{jk}$$
$$D^j = N_k A^{jk}$$

Gauge (coordinate) transformation

$$h^{\alpha\beta} \to h^{\alpha\beta} - \partial^\alpha \zeta^\beta - \partial^\beta \zeta^\alpha + \left(\partial_\mu \zeta^\mu \right) \eta^{\alpha\beta}$$

Stay within harmonic gauge as long as $\Box \zeta^\alpha = 0$

$$\zeta^0 = \frac{G}{c^3 R} \alpha(\tau, \boldsymbol{N}) + O(R^{-2})$$
$$\zeta^j = \frac{G}{c^3 R} \beta^j(\tau, \boldsymbol{N}) + O(R^{-2})$$

Far wave-zone fields & TT gauge

After a gauge transformation:

$$h^{00} \to \frac{4GM}{c^2R} + \frac{G}{c^4R}(C + \dot\alpha + N_j\dot\beta^j)$$

$$h^{0j} \to \frac{G}{c^4R}(D^j + \dot\beta^j + N^j\dot\alpha)$$

$$h^{jk} \to \frac{G}{c^4R}\left[A^{jk} + 2N^{(j}\dot\beta^{k)} + \delta^{jk}(\dot\alpha - N_m\dot\beta^m)\right]$$

To kill the h^{00} & h^{0j} terms:

$$\dot\beta^j + N^j\dot\alpha = -D^j = -N_mA^{jm}$$

$$N_j\dot\beta^j + \dot\alpha = -N_jD^j = -C$$

Insert β^j

$$h^{jk} \to \frac{G}{c^4R}\left[A^{jk} - 2\dot\alpha N^jN^k - 2N^{(j}N_nA^{k)n} + \delta^{jk}(2\dot\alpha + N_mN_nA^{mn})\right]$$

$$N_kh^{jk} = 0$$

Note that $h = h^k_k = \frac{G}{c^4R}\left[A + 4\dot\alpha + N_kN_nA^{kn}\right]$

Free to choose α so that h = 0: $\dot\alpha = -\frac{1}{4}(A + N_kN_nA^{kn})$

Final answer:

$$h^{jk} = \frac{G}{c^4R}A_{mn}\left[P^{jm}P^{kn} - \frac{1}{2}P^{jk}P^{mn}\right]$$

$$\equiv \frac{G}{c^4R}A^{jk}_{\rm TT}$$

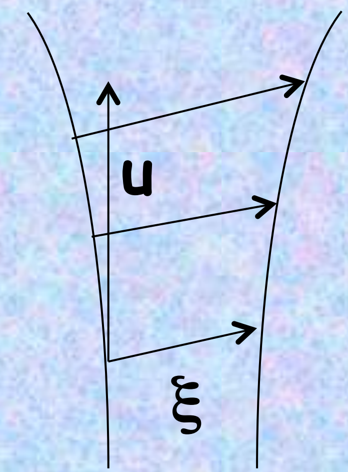
$$P^{jm} = \delta^{jm} - N^jN^m$$

$$P^{jm}P^{jn} = P^{mn}$$

$$P^{kk} = 2$$

Wave zone physics: Gravitational waves

Geodesic deviation: $$\frac{D^2\xi^\alpha}{ds^2} = -R^\alpha{}_{\beta\gamma\delta}u^\beta\xi^\gamma u^\delta$$



In the rest frame of an observer

$$\begin{aligned}\frac{d^2\xi^j}{dt^2} &= -c^2 R_{0j0k}\xi^k \\ &= \frac{1}{2}\partial_{\tau\tau}h^{jk}_{TT}\xi^k \\ &= \frac{G}{2c^4R}\ddot{A}^{jk}_{\mathrm{TT}}\xi^k\end{aligned}$$

$$\xi^j(t) = \xi^j(0) + \frac{1}{2}h^{jk}_{\mathrm{TT}}(t - R/c)\xi_k(0)$$

Wave zone physics: Gravitational waves

Wave in the z-direction

$$A^{jk}_{\rm TT} = \begin{pmatrix} A_+ & A_\times & 0 \\ A_\times & -A_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

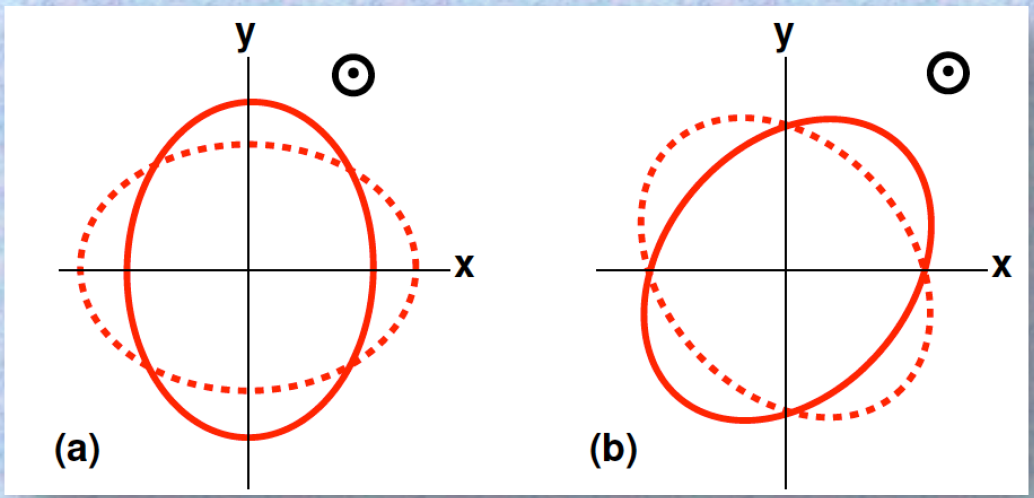
More generally: $\boldsymbol{e}_Z = \boldsymbol{N}$

$$A^{jk}_{\rm TT} = \left(e^j_X e^k_X - e^j_Y e^k_Y\right) A_+ + \left(e^j_X e^k_Y + e^j_Y e^k_X\right) A_\times$$

$$x(t) = x_0 + \frac{G}{2c^4 R}\Big(A_+ x_0 + A_\times y_0\Big)\,,$$

$$y(t) = y_0 + \frac{G}{2c^4 R}\Big(A_\times x_0 - A_+ y_0\Big)\,,$$

$$z(t) = z_0$$



Wave zone physics: The quadrupole formula

Requires **two** iterations of the relaxed Einstein equation:

$$
\begin{aligned}
h^{ij}_{2\mathcal{N}} &= \frac{4G}{c^4}\int \frac{\tau^{ij}(h_1)(t-|\boldsymbol{x}-\boldsymbol{x'}|/c,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|} d^3x' \\
&= \frac{2G}{rc^4}\ddot{\mathcal{I}}^{jk}(h_1)(\tau) + \frac{4G}{rc^4}\sum_{\ell=1}^{\infty}\frac{n^L}{c^\ell \ell!}\frac{d^\ell}{d\tau^\ell}\mathcal{M}^{jkL}(h_1)(\tau) \\
&= \frac{2G}{rc^4}\ddot{\mathcal{I}}^{jk}(h_1)(\tau) + O(c^{-5})
\end{aligned}
$$

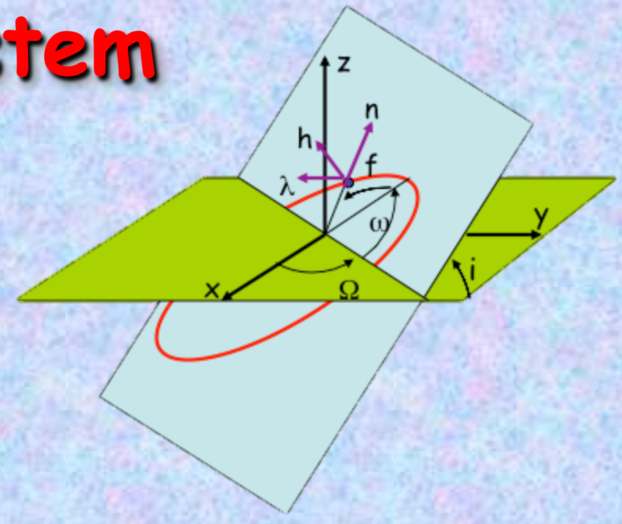
$$
\begin{aligned}
\mathcal{I}^{jk}(\tau) &:= \int_{\mathcal{M}} c^{-2}\tau^{00}(\tau,\boldsymbol{x'})x'^j x'^k \, d^3x' \\
&= \int_{\mathcal{M}} \rho^*(\tau,\boldsymbol{x'})x'^j x'^k \, d^3x' + O(c^{-2})
\end{aligned}
$$

For an N-body system

$$
\ddot{\mathcal{I}}^{ij} = 2\sum_A M_A v_A^{ij} - \sum_{A\neq B}\frac{GM_AM_B}{r_{AB}}n_{AB}^{ij}
$$

- By convention, the quadrupole formula is called the ``Newtonian"-order result
- Higher order PN corrections can be calculated by further iterating the relaxed equations
- 3 iterations needed for 1 & 1.5 PN order, 4 for 2PN order etc, but full details usually not needed

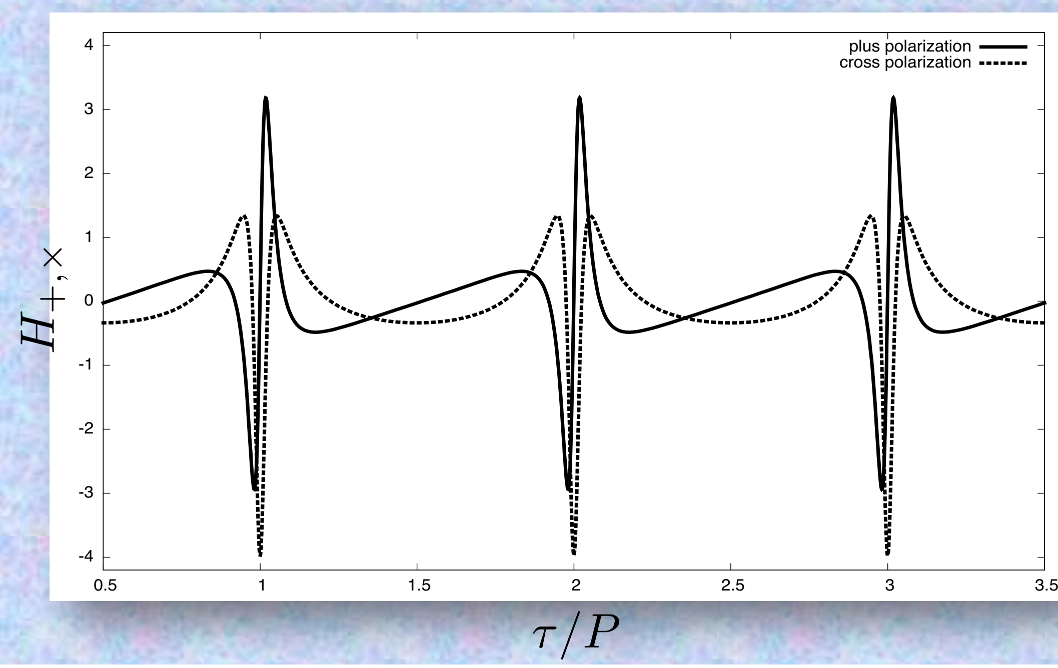
Waves from a binary system



$$h_{+(\times)} = \frac{2Gm\eta}{c^2 R}\left(\frac{Gm}{c^2 p}\right) H_{+(\times)}$$

$$H_+ = -(1+\cos^2\iota)\left[\cos(2\phi+2\omega) + \frac{5}{4}e\cos(\phi+2\omega) + \frac{1}{4}e\cos(3\phi+2\omega) + \frac{1}{2}e^2\cos 2\omega\right] + \frac{1}{2}e\sin^2\iota(\cos\phi+e),$$

$$H_\times = -2\cos\iota\left[\sin(2\phi+2\omega) + \frac{5}{4}e\sin(\phi+2\omega) + \frac{1}{4}e\sin(3\phi+2\omega) + \frac{1}{2}e^2\sin 2\omega\right]$$

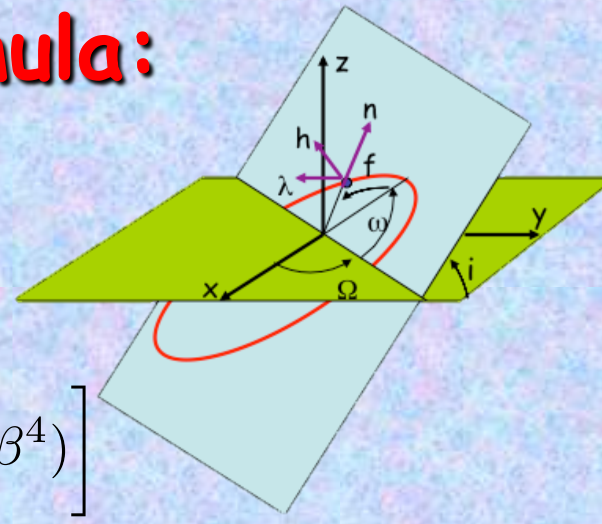


$$e = 0.7$$

$$\iota = 30^\circ$$

$$\omega = 45^\circ$$

Beyond the quadrupole formula:



For a binary system in a circular orbit:

$$h_{+,\times} = \frac{2\eta G m}{c^2 R}\beta^2 \Big[(1+2\pi\beta^3)H^{[0]}_{+,\times} + \Delta\beta H^{[1/2]}_{+,\times} + \beta^2 H^{[1]}_{+,\times} + \Delta\beta^3 H^{[3/2]}_{+,\times} + \beta^3 H^{\rm tail}_{+,\times} + O(\beta^4)\Big]$$

$$H^{[0]}_{\times} = -2C\sin 2\Psi,$$

$$H^{[1/2]}_{\times} = -\frac{3}{4}SC\sin\Psi + \frac{9}{4}SC\sin 3\Psi,$$

$$H^{[1]}_{\times} = \frac{1}{3}C\Big[(17-4C^2)-(13-12C^2)\eta\Big]\sin 2\Psi - \frac{8}{3}(1-3\eta)S^2C\sin 4\Psi,$$

$$H^{[3/2]}_{\times} = \frac{1}{96}SC\Big[(63-5C^2)-2(23-5C^2)\eta\Big]\sin\Psi - \frac{9}{64}SC\Big[(67-15C^2)-2(19-15C^2)\eta\Big]\sin 3\Psi + \frac{625}{192}(1-2\eta)S^3C\sin 5\Psi,$$

$$C = \cos\iota \qquad S = \sin\iota$$

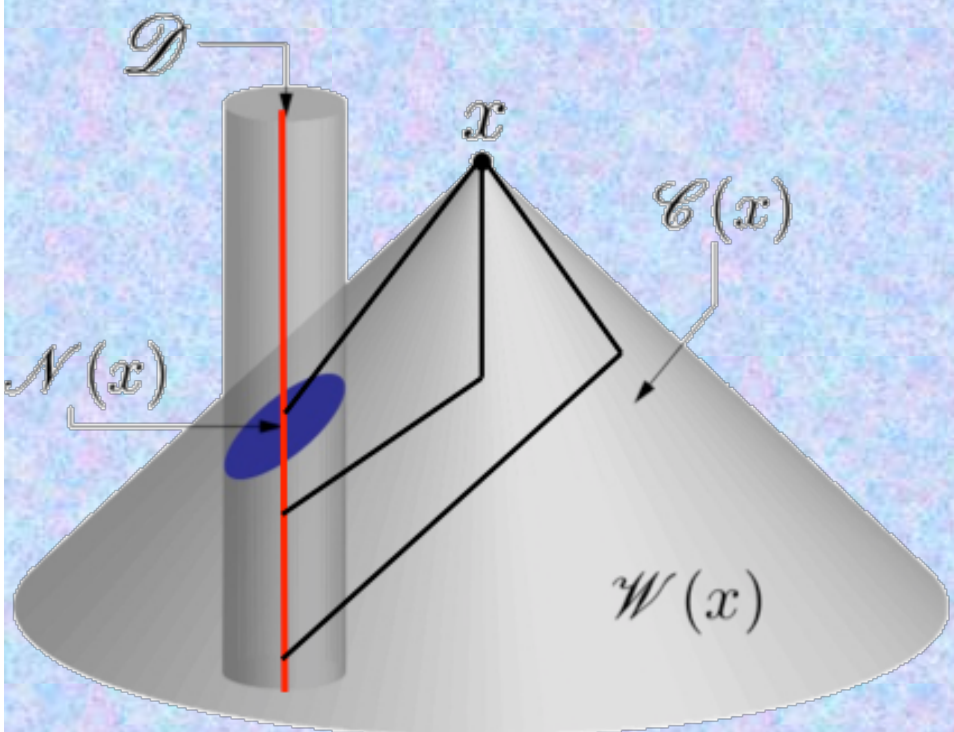
$$\beta = \left(\frac{Gm\Omega}{c^3}\right)^{1/3} \sim \frac{v}{c}, \quad m = M_1+M_2\,,\ \eta = \frac{M_1M_2}{(M_1+M_2)^2}\,,\ \Delta = \frac{M_1-M_2}{M_1+M_2}$$

Beyond the quadrupole formula:

$$h_{+,\times} = \frac{2\eta G m}{c^2 R}\beta^2 \Big[(1 + 2\pi\beta^3) H^{[0]}_{+,\times} + \Delta\beta H^{[1/2]}_{+,\times} + \beta^2 H^{[1]}_{+,\times} + \Delta\beta^3 H^{[3/2]}_{+,\times} + \beta^3 H^{\rm tail}_{+,\times} + O(\beta^4) \Big]$$

$$\beta = \left(\frac{Gm\Omega}{c^3}\right)^{1/3}$$

$$H^{[0]}_{\times} = -2C \sin 2\Psi \qquad H^{\rm tail}_{\times} = 8C\left[\gamma + \ln(4\Omega R/c)\right] \cos 2\Psi$$



Combine with the leading term:

$$\sin 2\Psi - 4\frac{Gm\Omega}{c^3}\ln(4\Omega R/c)\cos 2\Psi$$
$$= \sin\left[2\Omega\left(t - R/c - 2\frac{Gm}{c^3}\ln(4\Omega R/c)\right)\right]$$

Shapiro time delay

Gravitational-wave estimates

$$h^{jk}_{\rm TT} = \frac{2G}{c^4 R}\ddot{I}^{jk}_{\rm TT}$$

$$\sim \frac{GM}{c^2 R}(v_c/c)^2 \quad \sim 4.8\times 10^{-19}\left(\frac{M}{10\,M_\odot}\right)\left(\frac{1\,{\rm Mpc}}{R}\right)(v_c/c)^2$$

Binary system

$$(v_c/c)^2 \sim GM/c^2 a \sim (2\pi GM/c^3 P)^{2/3}$$

$$h_0 \simeq \frac{3.0\times 10^{-21}}{1-e^2}\left(\frac{\mathcal{M}}{20\,M_\odot}\right)^{5/3}\left(\frac{10\,{\rm ms}}{P}\right)^{2/3}\left(\frac{100\,{\rm Mpc}}{R}\right)$$

Chirp mass

$$\mathcal{M} \equiv \eta^{3/5} m = \left(\frac{m_1^3 m_2^3}{m}\right)^{1/5}$$

Deformed neutron star

$$h_0 \simeq 6.8\times 10^{-25}\left(\frac{\varepsilon}{10^{-6}}\right)\left(\frac{I_3}{1.6\times 10^{38}\ {\rm kg\,m^2}}\right)\left(\frac{10\ {\rm ms}}{P}\right)^2\left(\frac{1\ {\rm kpc}}{R}\right)$$

Supernova core collapse

$$h_0 \sim 4.8\times 10^{-21}\left(\frac{\varepsilon}{10^{-3}}\right)\left(\frac{M}{M_\odot}\right)\left(\frac{10\,{\rm kpc}}{R}\right)$$