Importance Sampling[1]

$$I = \int_a^b f(x)\,dx \qquad (1.1)$$

Introduce a positive definite sampling function g(x) and let

$$t = \frac{\int_a^x g(y)\,dy}{\int_a^b g(y)\,dy} = \frac{G(x;a)}{G(b;a)} \qquad (1.2)$$

With a positive definite g, G is single valued function that starts at 0 for x = a and ends at 1 for x =b. Thus t is a number between 0 and 1. Notice that x has move the the upper limit of the integral.

With this definition for the function x(t)

$$\mathrm{dt} = \frac{\mathrm{g(x)dx}}{G(b;a)} \qquad (1.3)$$

So that

$$I = G(b;a)\int_0^1 \frac{f(x(t))}{g(x(t))}\,dt \qquad (1.4)$$

This is discussed at length for analytically integral functions for which an immediate solution for x(t) is possible in Laurent.doc. the Laurent transform is for the 0 to ∞ range while an arc tangent transform is used for the -∞ to ∞ range.

Semi-infinite range

Write the integral

$$I = \int_0^\infty f(x)\,dx \quad (2.1)$$

Let

$$g(x) = \exp(-\alpha x) \quad (2.2)$$

[1] J. M. Hammersley and D. C. Handscomb, **Monte Carlo Methods**, Methune & Co. Ltd, London, John Wiley & Sons Inc, New York. pp. 57 -59

The region (a,b) in (1.2) becomes

$$G(\infty,0)=\frac{1}{\alpha} \quad (2.3)$$

The value of the integral given in (1.4) becomes

$$I=\frac{1}{\alpha}\int_0^1 \exp\left(\alpha x(t)\right) f\left(x(t)\right)dt \qquad (2.4)$$

The value of x(t) is needed. Equation (1.2) becomes

$$t=\frac{\int_0^x \exp(-\alpha y)\,dy}{\int_0^\infty \exp(-\alpha y)\,dy}=\frac{\left(\frac{1}{\alpha}\left(1-\exp(-\alpha x)\right)\right)}{\frac{1}{\alpha}}=\left(1-\exp(-\alpha x)\right) \qquad (2.5)$$

This solves as

$$\begin{aligned} t-1&=\exp(-\alpha x)\\ x&=\ln(1-t)/\alpha \end{aligned} \qquad (2.6)$$

Numerically

For t=1, x=ln(1-1)/α=∞, but computers do not handle ln(1-1) very well. In order to leave a few digits for the last term the ending point for the integral in t_{max} should be $(1\text{-}10^{-13})$. This means that the integration in t extends only to 13×2.3/α.

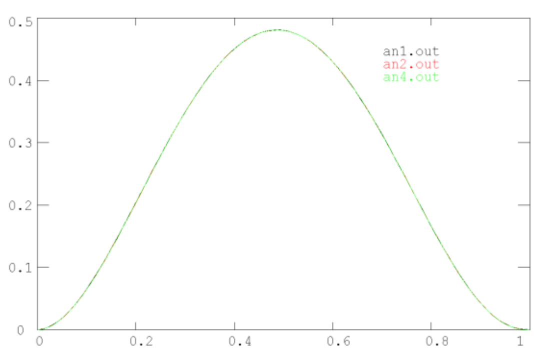

Figure 1 $r^2(t)\exp(-2r(t))$ versus t
100, 200, 400 pts are used toevaluatethe integral

AN 2.500000151518428E-01 100 pts
AN2 2.500000011811078E-01 200 PTS
AN4 2.500000000742313E-01 400 PTS
ANR 2.500000000566618E-01 Richardson's extrapolation
ANRF 2.500000005080060E-01 ± 4.443478404E-10 Fit of two points to AN, AN2, AN4 [..\Fittery\nlfit-r\StdDev\3ptLinFit.docx] Answer is ¼. The code is in SampleInt.zip

Solving for x(t)with an integrable function

The value of x(t) can be found by solving the equation

$$t \times G(a;b) - G(x;a) = 0 \qquad (3.1)$$

Newton's method, ..\optimization\solving\Newton.doc .htm, involves expanding the (3.1) as a function of x about x_0, setting the result equal to zero and solving for the next value of x

$$tG(a;b) - G(x_0,a) - (x_1 - x_0)G'(x_0,a) = 0 \quad (3.2)$$

So that

$$x_1 = x_0 - \frac{tG(a;b) - G(x_0,a)}{G'(x_0,a)} \qquad (3.3)$$

Note that G' is g so that the sequence

$$x_0 \to x_0 \frac{tG(a;b) - G(x_0,a)}{g(x_0)} \qquad (3.4)$$

can be iterated to find x(t). Note that this could take a lot of computer time if every value of $G(x_0;a)$ requires a revaluation of the integral in the numerator of (1.2). Normally, there would be a Lagrange interpolation of a single set of points, but this can introduce errors. An exact method involves making g(x) explicitly the straight line connecting a set of values $g(x_i)$.

Line connecting the points modification.

A very simple g(x) is the line connecting a group of points

$$g(x) = g(x_i) + (x - x_i)\frac{g(x_{i+1}) - g(x_i)}{x_{i+1} - x_i} \qquad x_i \le x \le x_{i+1} \qquad (3.5)$$

Let $y = x - x_{i-1}$ so that

$$\begin{aligned} G(x_0) &= 0 \\ G(x_i + y) &= G(x_i) + g(x_i)\int_0^t dy + \frac{g(x_{i+1}) - g(x_i)}{x_{i+1} - x_i}\int_0^t y\,dy \qquad (3.6) \\ &= G(x_i) + g(x_i) \times y + \frac{g(x_{i+1}) - g(x_i)}{x_{i+1} - x_i}\frac{y^2}{2} \end{aligned}$$

For $y = x_i - x_{i-1}$ the positive first term cancels half of the negative part of the second term leading to the seemingly linear

$$G(x_0)=0$$
$$G(x_{i+1})=G(x_i)+\frac{g(x_i)+g(x_{i+1})}{2_i}(x_{i+1}-x_i) \qquad (3.7)$$

The values in (3.7) are the **exact** values of $G(x_i)$ for the g(x) that is the line connecting the points $g(x_i)$. The values between these points are given exactly by (3.6).

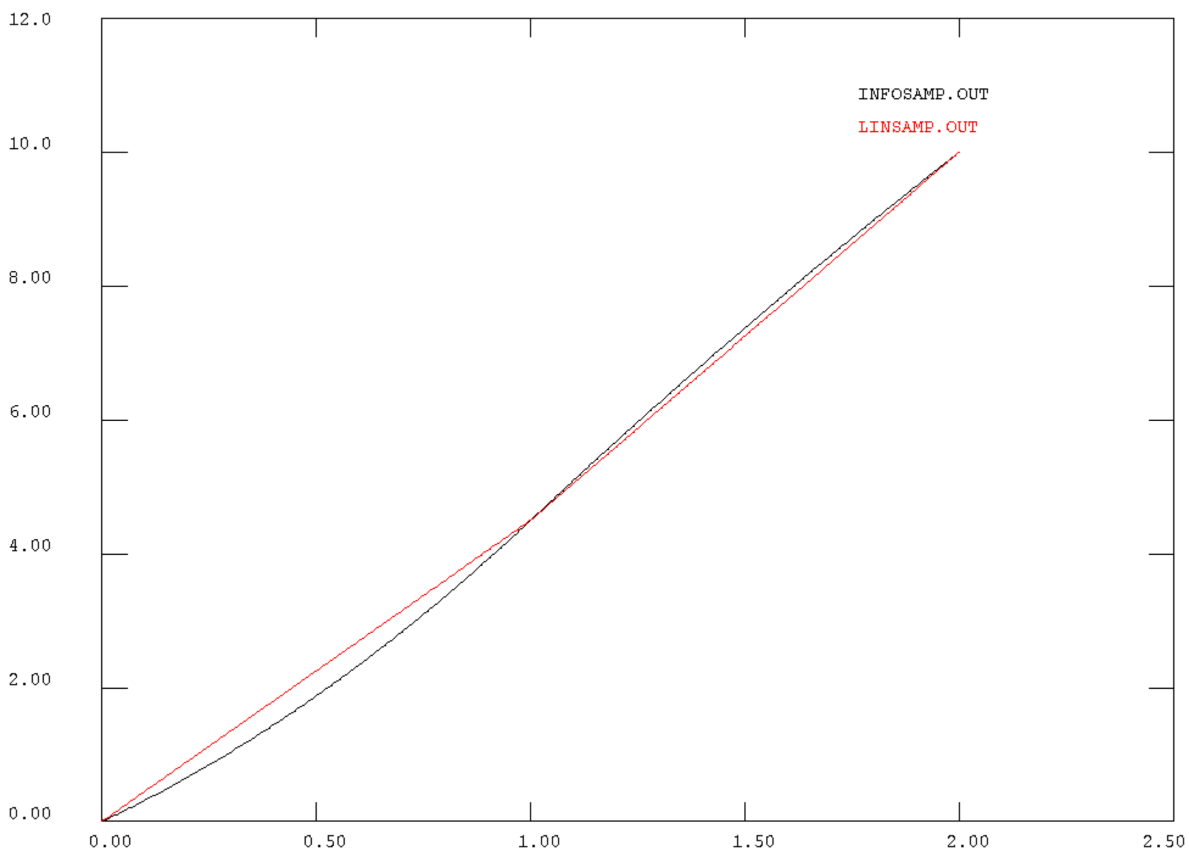

Figure 2 Integral of line connecting points (black). Linear interpolation of this same line. The g values are g(1)=3, g(2)=6,g(3)=5. The integral is below the linear interpolation for a positive g' and below it for a negative g'. Code is in infosamp.zip.

Solving for x(t)in line connecting the points modification

The value of $G(x_i)$ form an ascending series of values starting at 0 and ending at 1.

$$G(0)=0$$
$$G(i+1)=G(i)+\frac{g(i+1)+g(i)}{2}\left(X(i+1)-X(i)\right) \qquad (3.8)$$

The subroutine **LOCATE(tG(NMAX),G,NMAX,J)** ..\interpolation\Locate.doc returns a value J such that G(J)≤t≤G(J+1). In the region X(J) ≤ x ≤ X(J+1)

$$G(x.a) = G(J) + g(J)(x - X(J)) + \frac{1}{2}\frac{g(J+1) - g(J)}{X(J+1) - X(J)}(x - X(J))^2 \qquad (3.9)$$

Equation (1.2) becomes

$$tG(b;a) - G(J) - g(J)(x - X(J)) - \frac{1}{2}\frac{g(J+1) - g(J)}{X(J+1) - X(J)}(x - X(J))^2 = 0 \qquad (3.10)$$

Define

$$\begin{aligned} y &= x - X(J) \\ C &= tG(a,b) - G(J) \\ B &= -g\{J\} \\ A &= -\frac{1}{2}\frac{g(J+1) - g(J)}{X(J+1) - X(J)} \end{aligned} \qquad (3.11)$$

So that

$$Ay^2 + By + C = 0 \qquad (3.12)$$

This is a quadratic equation with a general solution given by (..\solving\Quadratic.doc).

$$y = -\frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A} \qquad (3.13)$$

C is greater than or equal to zero, since locate returned a J such that tG(a,b) > G(J). B is always less than 0, while the sign of A is unknown.

Equation (3.13) is numerically unsuitable since it will involve large cancellations. Following (..\solving\Quadratic.doc) rewrite (3.13) as

$$y = -\frac{B \mp \sqrt{B^2 - 4AC}}{2A} \times \frac{B \pm \sqrt{B^2 - 4AC}}{B \pm \sqrt{B^2 - 4AC}} = -\frac{B^2 - B^2 + 4AC}{2A\left(B \pm \sqrt{B^2 - 4AC}\right)} = \frac{-2C}{\left(B \pm B\sqrt{1 - 4AC / B^2}\right)}$$

(3.14)

The + sign yields y > 0. So

$$y = \frac{-2C}{B\left(1 + \sqrt{1 - 4AC / B^2}\right)} \qquad (3.15)$$

The largest value of C is G(J+1)-G(j) =(g(J+1)+g(J))(X(J+1)-X(J))/2 so that the largest value of $4AC/B^2$ is

$$
\begin{aligned}
\frac{4AC}{B^2} &= 4\times\frac{-1}{2}\frac{g(J+1)-g(J)}{X(J+1)-X(J)}\times\frac{\left(g(J+1)+g(J)\right)\left(X(J+1)-X(J)\right)}{2}\times\frac{1}{g^2\{J\}} \\
&= -\frac{g^2(J+1)-g^2(J)}{g^2(J)} \qquad (3.16) \\
&= 1-\frac{g^2(J+1)}{g^2(J)}
\end{aligned}
$$

This means that the most negative value of the argument of the square root in (3.15) is

$$1-4AC/B^2 > \frac{g^2(J+1)}{g^2(J)} \qquad (3.17)$$

Thus the value of y is never imaginary.

Summary

Find an arrangement of g(J)=g(x_J). Use equation (3.7) to find G(J). Then for values of t between 0 and 1, use locate(tG(NMAX),G,NMAX.J) to find the relevant J. Use (3.11) to define the terms in (3.15) which yields y(t). Finally

$$x(t) = x(J) + y(t) \qquad (3.18)$$