

Quadratic Integration

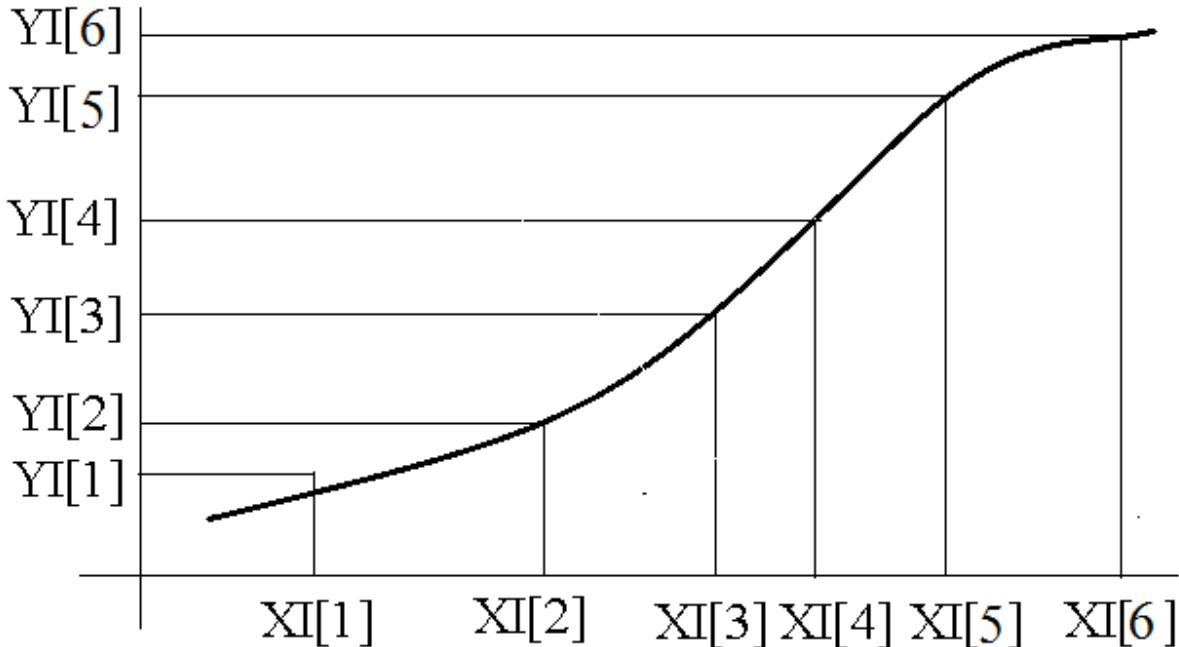


Figure 1 Values $YI[1], \dots, YI[4]$, at locations $XI[1], \dots, XI[4]$

The function described in (2) passes through the points $XI[2], YI[2]$ and $XI[3], YI[3]$, but not through $XI[1], YI[1]$ and $XI[4], YI[4]$. The first derivatives at the mid points between the data points have errors proportional to the size of the intervals squared (equation **Error! Reference source not found.**). The difference between the first derivatives in the first and last region gives an accurate estimate of the second derivative in the middle region (equation (24)). The value of the function in the middle region is accurate to third order in the difference cubed and as shown in Figure 3 and Figure 4 extrapolates fairly well.

Expand the function about $(x_3+x_4)/2$

Expand through fourth order

$$(1) \quad f(x) = f\left(\frac{x_3 + x_4}{2}\right) + \left(x - \frac{x_3 + x_4}{2}\right) f'\left(\frac{x_3 + x_4}{2}\right) + \frac{1}{2} \left(x - \frac{x_3 + x_4}{2}\right)^2 f''\left(\frac{x_3 + x_4}{2}\right) + \frac{1}{6} \left(x - \frac{x_3 + x_4}{2}\right)^3 f'''\left(\frac{x_3 + x_4}{2}\right) + \frac{1}{24} \left(x - \frac{x_3 + x_4}{2}\right)^4 f^{iv}\left(\frac{x_3 + x_4}{2}\right)$$

Define

$$(3) \quad F(x) = F(x_3) + \int_{x_3}^x f(y) dy$$

Let

$$t = x - \frac{x_3 + x_4}{2} \quad (4)$$

So that

$$\begin{aligned} F(x_4) &= F(x_3) + \int_{-\frac{x_4 - x_3}{2}}^{\frac{x_4 - x_3}{2}} f\left(\frac{x_3 + x_4}{2}\right) dt + \int_{-\frac{x_4 - x_3}{2}}^{\frac{x_4 - x_3}{2}} t f'\left(\frac{x_3 + x_4}{2}\right) dt \\ &\quad + \frac{1}{2} \int_{-\frac{x_4 - x_3}{2}}^{\frac{x_4 - x_3}{2}} t^2 f''\left(\frac{x_3 + x_4}{2}\right) dt + \frac{1}{6} \int_{-\frac{x_4 - x_3}{2}}^{\frac{x_4 - x_3}{2}} t^3 f'''\left(\frac{x_3 + x_4}{2}\right) dt \quad (5) \\ &\quad + \frac{1}{24} \int_{-\frac{x_4 - x_3}{2}}^{\frac{x_4 - x_3}{2}} t^4 f^{iv}\left(\frac{x_3 + x_4}{2}\right) dt \end{aligned}$$

The odd terms integrate to zero across these regions so that

$$\begin{aligned} F(x_4) &= F(x_3) + (x_4 - x_3) f\left(\frac{x_3 + x_4}{2}\right) + \frac{1}{2} \times \frac{2}{3} \left(\frac{x_4 - x_3}{2}\right)^3 f''\left(\frac{x_3 + x_4}{2}\right) + \frac{1}{24} \times \frac{2}{5} \left(\frac{x_4 - x_3}{2}\right)^5 f^{iv}\left(\frac{x_3 + x_4}{2}\right) \quad (6) \\ &= F(x_3) + (x_4 - x_3) f\left(\frac{x_3 + x_4}{2}\right) + \frac{1}{24} (x_4 - x_3)^3 f''\left(\frac{x_3 + x_4}{2}\right) + \frac{1}{1920} (x_4 - x_3)^5 f^{iv}\left(\frac{x_3 + x_4}{2}\right) \end{aligned}$$

Values of f , f'' , and f^{iv} at $(x_3+x_4)/2$ are needed

Analytic Example

$$\begin{aligned} f(x) &= \exp(-x^2) \\ f'(x) &= -2x \exp(-x^2) \\ f''(x) &= (-2 + 4x^2) \exp(-x^2) \\ f'''(x) &= \{(8x - 2 + 4x^2) - (-2 + 4x^2)2x\} \exp(-x^2) \quad (7) \\ &= \{-2 + 12x + 4x^2 - 8x^3\} \exp(-x^2) \\ f^{iv}(x) &= \{(12 + 8x - 24x^2) - 2x(-2 + 12x + 4x^2 - 8x^3)\} \exp(-x^2) \\ &= \{12 + 12x - 48x^2 - 8x^3 + 16x^4\} \exp(-x^2) \end{aligned}$$

The second and fourth derivatives should be checked numerically.

[..\Derivatives\GDC2nd.zip](#) in [..../Derivatives/NumericalDerivatives.docx#Numeircal2nd](#)

[GderivC24.zip](#)

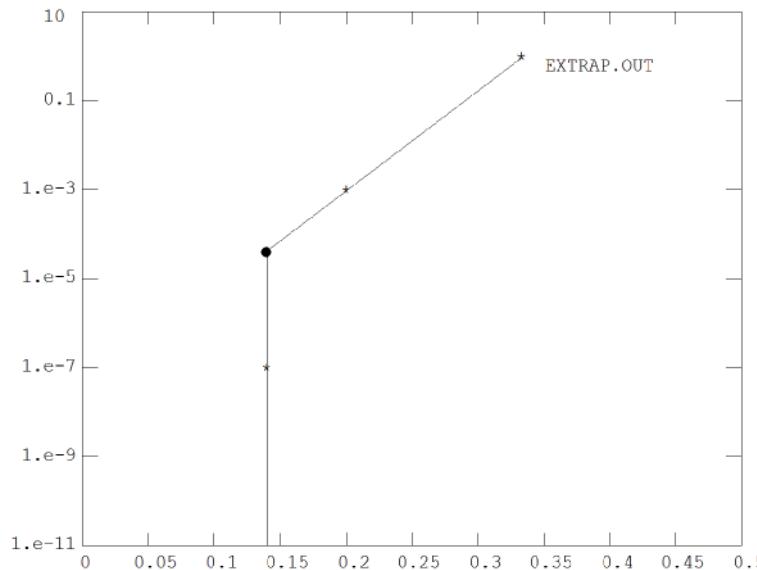


Figure 2 Simulated values of the h^3 and h^5 terms as a function of $1/\text{Pow}$

In figure 2 the term proportional to h^3 is at 0.133, the one proportional to h^5 is at 0.2 and the first missing term at h^7 is found by extrapolating to 0.14. This gives a best estimate of the error in the integral.

$$a + b(1/3) = \log(T3)$$

$$a + b(1/5) = \log(T5)$$

$$b = (\log(T3) - \log(T5)) / (1/3 - 1/5) \quad (8)$$

$$a = \log(T3) - b(1/3)$$

$$\log(T7) = a + b(1/7)$$

Note that (8) gives the magnitude. The sign will be positive if T3 and T5 are positive, negative if these are both negative and then in accord with the two signs.

In general the error will be $\min(T7, T5, T3)$. In summing over regions, the error estimate will be

$$\max\left(\sqrt{\sum err^2}, \sum err\right)$$

ENTER X,DELTA TO CHECK FOURTH DERIV O 0 TO STOP

3 1D-7

ANALYTIC 3.0000000000000000 0.0858932236443290

NUMERICAL 3.0000000000000000 0.0858932236443290

ENTER X,DELTA TO CHECK FOURTH DERIV O 0 TO STOP

3.7

1D-7

ANALYTIC 3.7000000000000002 0.0022591934880722

NUMERICAL 3.7000000000000002 0.0022591934880722

Run a Lagrange Polynomial through the six points

Expand the function about $(x_1+x_2)/2$

$$\begin{aligned} f(x_1) &= f\left(\frac{x_1+x_2}{2}\right) + \left(x_1 - \frac{x_1+x_2}{2}\right) f'\left(\frac{x_1+x_3}{2}\right) + \frac{1}{2}\left(x_1 - \frac{x_1+x_2}{2}\right)^2 f''\left(\frac{x_1+x_2}{2}\right) \\ &\quad + \frac{1}{6}\left(x_1 - \frac{x_1+x_2}{2}\right)^3 f'''\left(\frac{x_1+x_2}{2}\right) + \frac{1}{24}\left(x_1 - \frac{x_1+x_2}{2}\right)^3 f^{iv}\left(\frac{x_1+x_2}{2}\right) \\ &= f\left(\frac{x_1+x_2}{2}\right) - \left(\frac{x_2-x_1}{2}\right) f'\left(\frac{x_1+x_3}{2}\right) + \frac{1}{2}\left(\frac{x_2-x_1}{2}\right)^2 f''\left(\frac{x_1+x_2}{2}\right) \\ &\quad - \frac{1}{6}\left(\frac{x_2-x_1}{2}\right)^3 f'''\left(\frac{x_1+x_2}{2}\right) + \frac{1}{24}\left(\frac{x_2-x_1}{2}\right)^3 f^{iv}\left(\frac{x_1+x_2}{2}\right) \end{aligned} \quad (9)$$

$$\begin{aligned} f(x_2) &= f\left(\frac{x_1+x_2}{2}\right) + \left(\frac{x_2-x_1}{2}\right) f'\left(\frac{x_1+x_2}{2}\right) + \frac{1}{2}\left(\frac{x_2-x_1}{2}\right)^2 f''\left(\frac{x_1+x_2}{2}\right) \\ &\quad + \frac{1}{6}\left(\frac{x_2-x_1}{2}\right)^3 f'''\left(\frac{x_1+x_2}{2}\right) + \frac{1}{24}\left(\frac{x_2-x_1}{2}\right)^3 f^{iv}\left(\frac{x_1+x_2}{2}\right) \end{aligned} \quad (10)$$

Subtract (9) from (10)

$$f(x_2) - f(x_1) = (x_2 - x_1) f'\left(\frac{x_1+x_2}{2}\right) + \frac{2}{6}\left(\frac{x_2-x_1}{2}\right)^3 f'''\left(\frac{x_1+x_2}{2}\right) \quad (11)$$

Expand the function about $(x_3+x_4)/2$

$$\begin{aligned} f(x_3) &= f\left(\frac{x_3+x_4}{2}\right) - \left(\frac{x_4-x_3}{2}\right) f'\left(\frac{x_3+x_4}{2}\right) + \frac{1}{2}\left(\frac{x_4-x_3}{2}\right)^2 f''\left(\frac{x_3+x_4}{2}\right) \\ &\quad - \frac{1}{6}\left(\frac{x_4-x_3}{2}\right)^3 f'''\left(\frac{x_3+x_4}{2}\right) + \frac{1}{24}\left(\frac{x_4-x_3}{2}\right)^3 f^{iv}\left(\frac{x_3+x_4}{2}\right) \end{aligned} \quad (12)$$

$$\begin{aligned} f(x_4) &= f\left(\frac{x_3+x_4}{2}\right) + \left(\frac{x_4-x_3}{2}\right) f'\left(\frac{x_3+x_4}{2}\right) + \frac{1}{2}\left(\frac{x_4-x_3}{2}\right)^2 f''\left(\frac{x_3+x_4}{2}\right) \\ &\quad + \frac{1}{6}\left(\frac{x_4-x_3}{2}\right)^3 f'''\left(\frac{x_3+x_4}{2}\right) + \frac{1}{24}\left(\frac{x_4-x_3}{2}\right)^3 f^{iv}\left(\frac{x_3+x_4}{2}\right) \end{aligned} \quad (13)$$

Subtract (12) from (13)

$$f(x_4) - f(x_3) = (x_4 - x_3) f' \left(\frac{x_3 + x_4}{2} \right) + \frac{2}{6} \left(\frac{x_4 - x_3}{2} \right)^3 f''' \left(\frac{x_3 + x_4}{2} \right) \quad (14)$$

Expand the function about $(x_5+x_6)/2$

$$\begin{aligned} f(x_5) &= f \left(\frac{x_5 + x_6}{2} \right) - \left(\frac{x_6 - x_5}{2} \right) f' \left(\frac{x_5 + x_6}{2} \right) + \frac{1}{2} \left(\frac{x_6 - x_5}{2} \right)^2 f'' \left(\frac{x_5 + x_6}{2} \right) \\ &\quad + \frac{1}{6} \left(\frac{x_6 - x_5}{2} \right)^3 f''' \left(\frac{x_5 + x_6}{2} \right) - \frac{1}{24} \left(\frac{x_6 - x_5}{2} \right)^3 f^{iv} \left(\frac{x_5 + x_6}{2} \right) \end{aligned} \quad (15)$$

$$\begin{aligned} f(x_6) &= f \left(\frac{x_5 + x_6}{2} \right) + \left(\frac{x_6 - x_5}{2} \right) f' \left(\frac{x_5 + x_6}{2} \right) + \frac{1}{2} \left(\frac{x_6 - x_5}{2} \right)^2 f'' \left(\frac{x_5 + x_6}{2} \right) \\ &\quad + \frac{1}{6} \left(\frac{x_6 - x_5}{2} \right)^3 f''' \left(\frac{x_5 + x_6}{2} \right) + \frac{1}{24} \left(\frac{x_6 - x_5}{2} \right)^3 f^{iv} \left(\frac{x_5 + x_6}{2} \right) \end{aligned} \quad (16)$$

Subtract (15) from (16)

$$f(x_6) - f(x_5) = (x_6 - x_5) f' \left(\frac{x_5 + x_6}{2} \right) + \frac{2}{6} \left(\frac{x_6 - x_5}{2} \right)^3 f''' \left(\frac{x_5 + x_6}{2} \right) \quad (17)$$

Expand the first and third derivatives of the function about $(x_3+x_4)/2$

$$\begin{aligned} f'(x) &= f' \left(\frac{x_3 + x_4}{2} \right) + \left(x - \frac{x_3 + x_4}{2} \right) f'' \left(\frac{x_3 + x_4}{2} \right) + \frac{1}{2} \left(x - \frac{x_3 + x_4}{2} \right)^2 f''' \left(\frac{x_3 + x_4}{2} \right) \\ &\quad + \frac{1}{6} \left(x - \frac{x_3 + x_4}{2} \right)^3 f^{iv} \left(\frac{x_3 + x_4}{2} \right) \end{aligned} \quad (18)$$

$$f'''(x) = f''' \left(\frac{x_3 + x_4}{2} \right) + \left(x - \frac{x_3 + x_4}{2} \right) f^{iv} \left(\frac{x_3 + x_4}{2} \right) \quad (19)$$

Use these in (11)

$$\begin{aligned}
f(x_2) - f(x_1) &= (x_2 - x_1) \left\{ f' \left(\frac{x_3 + x_4}{2} \right) + \left(\frac{x_1 + x_2}{2} - \frac{x_3 + x_4}{2} \right) f'' \left(\frac{x_3 + x_4}{2} \right) \right. \\
&\quad \left. + \frac{1}{2} \left(\frac{x_1 + x_2}{2} - \frac{x_3 + x_4}{2} \right)^2 f''' \left(\frac{x_3 + x_4}{2} \right) \right. \\
&\quad \left. + \frac{1}{6} \left(\frac{x_1 + x_2}{2} - \frac{x_3 + x_4}{2} \right)^3 f^{iv} \left(\frac{x_3 + x_4}{2} \right) \right\} \\
&\quad + \frac{2}{6} \left(\frac{x_2 - x_1}{2} \right)^3 \left\{ f''' \left(\frac{x_3 + x_4}{2} \right) + \left(\frac{x_1 + x_2}{2} - \frac{x_3 + x_4}{2} \right) f^{iv} \left(\frac{x_3 + x_4}{2} \right) \right\} \\
&= (x_2 - x_1) f' \left(\frac{x_3 + x_4}{2} \right) \\
&\quad + (x_2 - x_1) \left(\frac{x_1 + x_2}{2} - \frac{x_3 + x_4}{2} \right) f'' \left(\frac{x_3 + x_4}{2} \right) \\
&\quad + \left\{ (x_2 - x_1) \frac{1}{2} \left(\frac{x_1 + x_2}{2} - \frac{x_3 + x_4}{2} \right)^2 + \frac{2}{6} \left(\frac{x_2 - x_1}{2} \right)^3 \right\} f''' \left(\frac{x_3 + x_4}{2} \right) \\
&\quad + \left\{ (x_2 - x_1) \frac{1}{6} \left(\frac{x_1 + x_2}{2} - \frac{x_3 + x_4}{2} \right)^3 + \frac{2}{6} \left(\frac{x_2 - x_1}{2} \right)^3 \left(\frac{x_1 + x_2}{2} - \frac{x_3 + x_4}{2} \right) \right\} f^{iv} \left(\frac{x_3 + x_4}{2} \right) \quad (20)
\end{aligned}$$

And in (17)

$$\begin{aligned}
f(x_6) - f(x_5) &= (x_6 - x_5) \left\{ f' \left(\frac{x_3 + x_4}{2} \right) + \left(\frac{x_5 + x_6}{2} - \frac{x_3 + x_4}{2} \right) f'' \left(\frac{x_3 + x_4}{2} \right) \right. \\
&\quad \left. + \frac{1}{2} \left(\frac{x_5 + x_6}{2} - \frac{x_3 + x_4}{2} \right)^2 f''' \left(\frac{x_3 + x_4}{2} \right) \right\} \\
&\quad + \frac{1}{6} \left(\frac{x_5 + x_6}{2} - \frac{x_3 + x_4}{2} \right)^3 f^{iv} \left(\frac{x_3 + x_4}{2} \right) \\
&\quad + \frac{2}{6} \left(\frac{x_5 - x_6}{2} \right)^3 \left\{ f''' \left(\frac{x_3 + x_4}{2} \right) + \left(\frac{x_5 + x_6}{2} - \frac{x_3 + x_4}{2} \right) f^{iv} \left(\frac{x_3 + x_4}{2} \right) \right\}
\end{aligned}$$

And the third derivative as

Evaluate (18) at $(x_1+x_2)/2$ and $(x_3+x_4)/2$

$$\begin{aligned}
f' \left(\frac{x_1 + x_2}{2} \right) &= f' \left(\frac{x_2 + x_3}{2} \right) + \left(\frac{x_1 + x_2}{2} - \frac{x_3 + x_2}{2} \right) f'' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{2} \left(\frac{x_1 + x_2}{2} - \frac{x_3 + x_2}{2} \right) f''' \left(\frac{x_2 + x_3}{2} \right) \quad (21) \\
f' \left(\frac{x_3 + x_4}{2} \right) &= f' \left(\frac{x_2 + x_3}{2} \right) + \left(\frac{x_3 + x_4}{2} - \frac{x_3 + x_2}{2} \right) f'' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{2} \left(\frac{x_3 + x_4}{2} - \frac{x_3 + x_2}{2} \right) f''' \left(\frac{x_2 + x_3}{2} \right)
\end{aligned}$$

Simplify the parenthesis

$$\begin{aligned} f' \left(\frac{x_1 + x_2}{2} \right) &= f' \left(\frac{x_2 + x_3}{2} \right) + \left(\frac{x_1 - x_3}{2} \right) f'' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{2} \left(\frac{x_1 - x_3}{2} \right)^2 f''' \left(\frac{x_2 + x_3}{2} \right) \\ f' \left(\frac{x_3 + x_4}{2} \right) &= f' \left(\frac{x_2 + x_3}{2} \right) + \left(\frac{x_4 - x_2}{2} \right) f'' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{2} \left(\frac{x_4 - x_2}{2} \right)^2 f''' \left(\frac{x_2 + x_3}{2} \right) \end{aligned} \quad (22)$$

Subtract line 1 from line 2

$$f' \left(\frac{x_3 + x_4}{2} \right) - f' \left(\frac{x_1 + x_2}{2} \right) = \frac{1}{2} ((x_3 + x_4) - (x_1 + x_2)) f'' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{8} ((x_4 - x_2)^2 - (x_3 - x_1)^2) f''' \left(\frac{x_2 + x_3}{2} \right) \quad (23)$$

Or

$$f'' \left(\frac{x_2 + x_3}{2} \right) = \frac{2 \left(f' \left(\frac{x_3 + x_4}{2} \right) - f' \left(\frac{x_1 + x_2}{2} \right) \right)}{((x_3 + x_4) - (x_1 + x_2))} - \frac{1}{4} \frac{((x_4 - x_2)^2 - (x_3 - x_1)^2)}{((x_3 + x_4) - (x_1 + x_2))} f''' \left(\frac{x_2 + x_3}{2} \right) \quad (24) \leftarrow$$

Write equation (2) for points 2 and 3

$$\begin{aligned} f(x_2) &= f \left(\frac{x_2 + x_3}{2} \right) + \left(x_2 - \frac{x_3 + x_2}{2} \right) f' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{2} \left(x_2 - \frac{x_3 + x_2}{2} \right)^2 f'' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{6} \left(x_2 - \frac{x_3 + x_2}{2} \right)^3 f''' \left(\frac{x_2 + x_3}{2} \right) \\ f(x_3) &= f \left(\frac{x_2 + x_3}{2} \right) + \left(x_3 - \frac{x_3 + x_2}{2} \right) f' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{2} \left(x_3 - \frac{x_3 + x_2}{2} \right)^2 f'' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{6} \left(x_3 - \frac{x_3 + x_2}{2} \right)^3 f''' \left(\frac{x_2 + x_3}{2} \right) \end{aligned} \quad (25)$$

Simplify the parenthesis

$$\begin{aligned} f(x_2) &= f \left(\frac{x_2 + x_3}{2} \right) + \left(\frac{x_2 - x_3}{2} \right) f' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{2} \left(\frac{x_2 - x_3}{2} \right)^2 f'' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{6} \left(\frac{x_2 - x_3}{2} \right)^3 f''' \left(\frac{x_2 + x_3}{2} \right) \\ f(x_3) &= f \left(\frac{x_2 + x_3}{2} \right) + \left(\frac{x_3 - x_2}{2} \right) f' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{2} \left(\frac{x_3 - x_2}{2} \right)^2 f'' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{6} \left(\frac{x_3 - x_2}{2} \right)^3 f''' \left(\frac{x_2 + x_3}{2} \right) \end{aligned} \quad (26)$$

Add the two lines in (26). The f''' term cancels.

$$f(x_2) + f(x_3) = 2f \left(\frac{x_2 + x_3}{2} \right) + \left(\frac{x_3 - x_2}{2} \right)^2 f'' \left(\frac{x_2 + x_3}{2} \right) \quad (27)$$

$$f \left(\frac{x_2 + x_3}{2} \right) = \frac{f(x_2) + f(x_3)}{2} - \frac{1}{2} \left(\frac{x_3 - x_2}{2} \right)^2 f'' \left(\frac{x_2 + x_3}{2} \right) \quad (28) \leftarrow$$

Assume that the term containing f''' can always be taken to be zero

The two values of f' are given by **Error! Reference source not found.** and **Error! Reference source not found.**

$$f' \left(\frac{x_3 + x_4}{2} \right) = \frac{f(x_4) - f(x_3)}{x_4 - x_3} \quad (29)$$

$$f' \left(\frac{x_1 + x_2}{2} \right) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad (30)$$

Summary

In the region between x_2 and x_3

$$xm = (x_2 + x_3)/2$$

$$\begin{aligned} f(x) &= a + b(x - x_m) + c(x - x_m)^2 \\ f'(x) &= b + 2c(x - x_m) \end{aligned}$$

$$fp34 = (dat(4) - dat(3)) / (x(4) - x(3))$$

$$fp23 = (dat(3) - dat(2)) / (x(3) - x(2))$$

$$fp12 = (dat(2) - dat(1)) / (x(2) - x(1))$$

$$fpp23 = 2 * (fp34 - fp12) / ((x_3 + x_4) - (x_1 + x_2))$$

$$F23 = (dat(2) + dat(3)) / 2 - (x_3 - x_2) * * 2 * Fpp23 / 2$$

$$a = F23$$

$$b = Fp12$$

$$c = Fpp23 / 2$$

```
FUNCTION QUAD(XI,YI,X,DQUADDX)
IMPLICITREAL*8 (A-H,O-Z)
DIMENSION XI(4),YI(4)
B=(YI(3)-YI(2))/(XI(3)-XI(2))
FP12=(YI(2)-YI(1))/(XI(2)-XI(1))
FP34=(YI(4)-YI(3))/(XI(4)-XI(3))
C=(FP34-FP12)/(XI(3)+XI(4)-XI(1)-XI(2)) !FPP23/2
A=(YI(2)+YI(3))/2-(XI(3)-XI(2))**2*C/4
```

$$f \left(\frac{x_2 + x_3}{2} \right) = \frac{f(x_2) + f(x_3)}{2} - \frac{1}{2} \left(\frac{x_3 - x_2}{2} \right)^2 f'' \left(\frac{x_2 + x_3}{2} \right) \quad (31)$$

$$XM = X - (XI(2) + XI(3)) / 2$$

$$DQUADDX = B + 2 * XM * C$$

$$QUAD = A + XM * (B + XM * C)$$

RETURN

END

```
function Quad(xi,yi:TALAG4;x:double;var dQuaddx:double):double;
var
  a,b,c,xm,fp12,fp34,fpp23:double;
begin
  b:=(yi[3]-yi[2])/(xi[3]-xi[2]);
```

```

fp12:=(yi[2]-yi[1])/(xi[2]-xi[1]);
fp34:=(yi[4]-yi[3])/(xi[4]-xi[3]);
c:=(fp34-fp12)/(xi[3]+xi[4]-xi[1]-xi[2]);
a:=(yi[2]+yi[3])/2-sqr(xi[3]-xi[2])*c/4;
xm:=(xi[2]+xi[3])/2;
dQuaddx:=b+2*(x-xm)*c;
Result:=a+(x-xm)*(b+(x-xm)*c);
end;

```

Beginning Region.

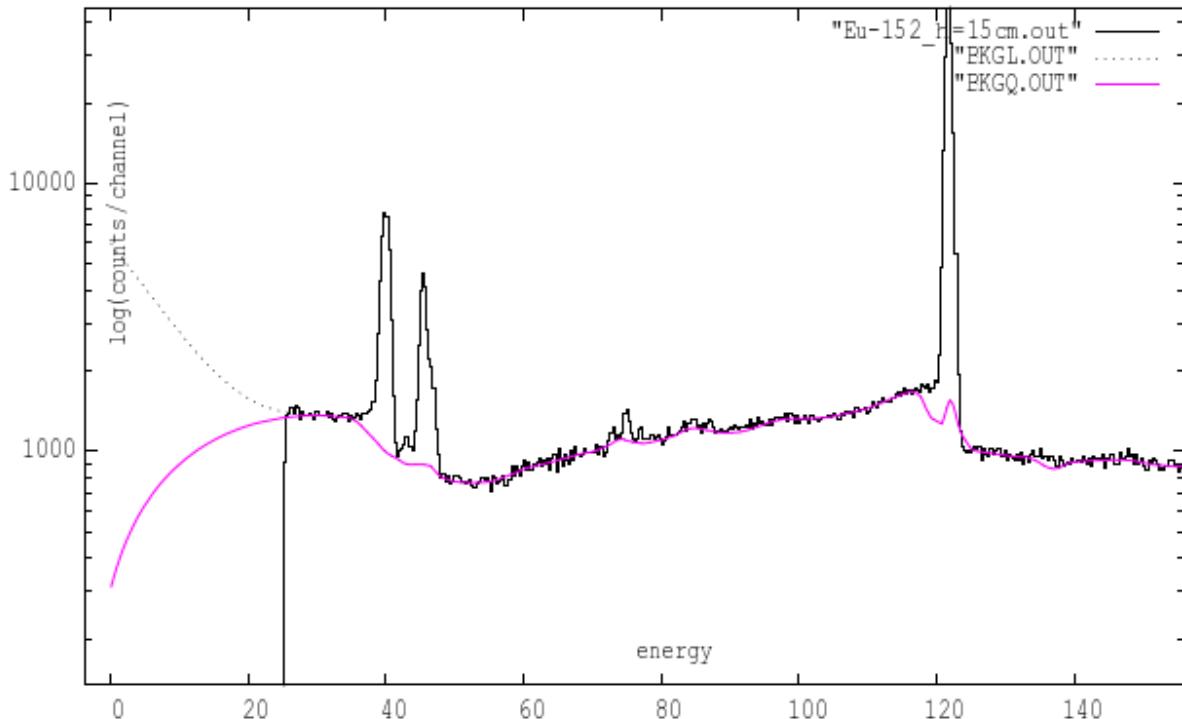


Figure 3 The fit is from 25.05 to 2526.47. The nuclides were fitted, but are not shown.

The values from the origin to 25.05 are outside the fit. The falling blue line is from equation (2) which is centered between the second and third fitted point. The rising dotted curve is from the Lagrange polynomial.

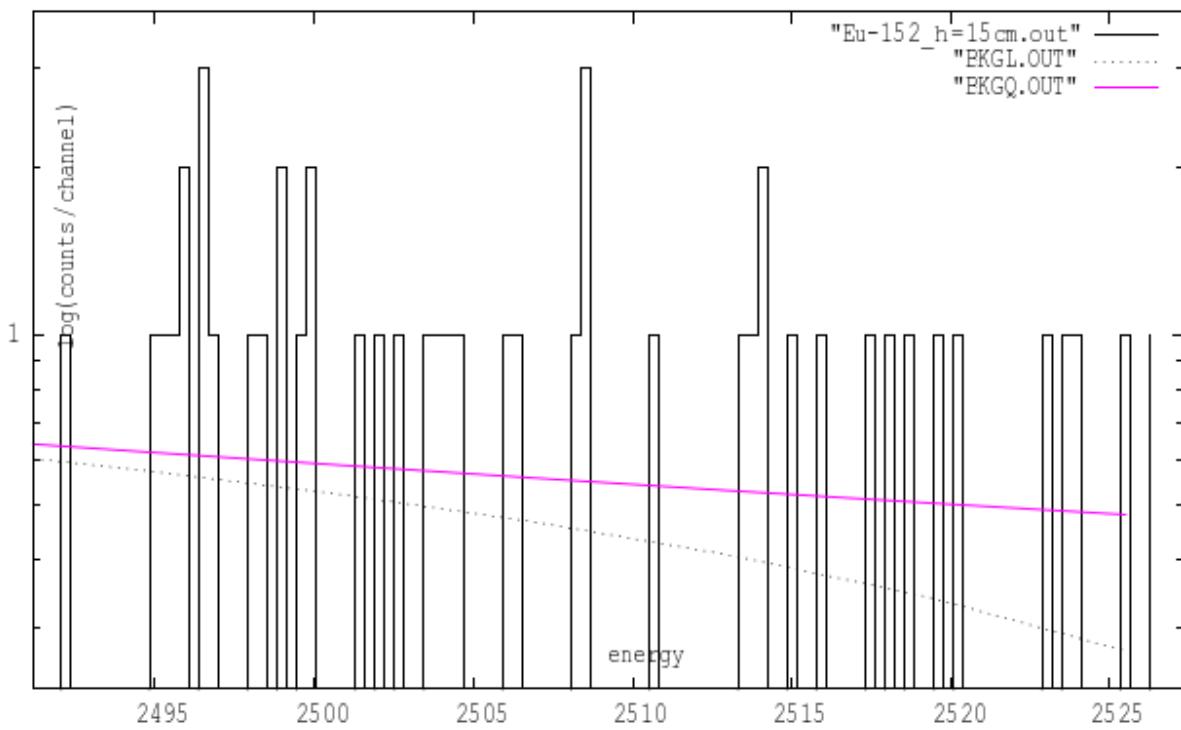


Figure 4 Upper end of fit.

The flatter blue line is from equation (2). The dotted lower line is from the Lagrange polynomials. The last fitted point is at about 2505. The blue line is actually centered before the last point at 2505, while the dotted line goes through this point.

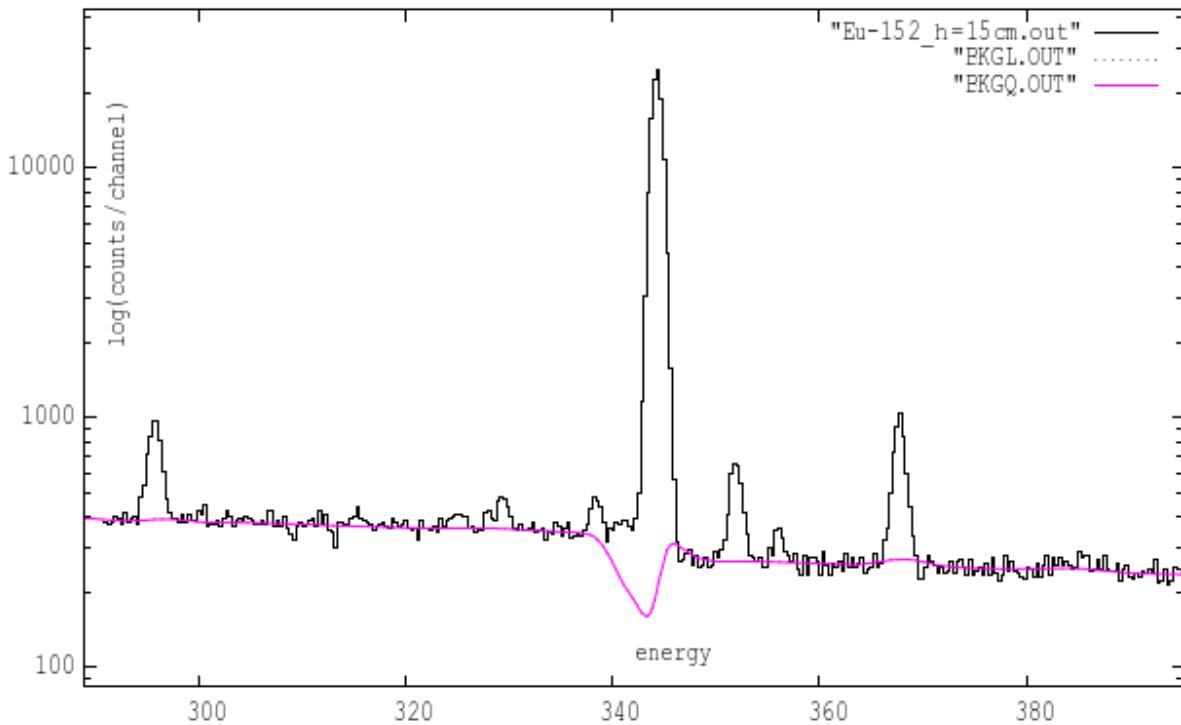


Figure 5 the 344 KeV peak is a bit off. the continuum approximations are not different to graphical accuracy.

