Trap rule error in periodic functions

Any function that has all derivatives at its beginning, t=0, equal to all derivatives at its end, t=T, can be expanded as a periodic function.

A periodic function can always be expanded as

$$f(t) = \sum_{j=0}^{\infty} c_j e^{2\pi i \frac{jt}{T}} \qquad (1)$$

Note that c_j is complex and that

$$f(t+T) = \sum_{j=0}^{\infty} c_j e^{2\pi i j \frac{t+T}{T}} = \sum_{j=0}^{\infty} c_j e^{2\pi i j} e^{2\pi i j(t/T)} = f(t) \qquad (2)$$

Since the first exponential is always 1. The integral from 0 to T of f is

$$\int_0^T f(t)dt = \sum_{j=0}^{\infty} c_j \int_0^T e^{2\pi i j(t/T)} dx = \sum_{j=0}^{\infty} c_j T \left(\delta_{0,j} + \frac{e^{2\pi i j} - 1}{2\pi i j} \right) = c_0 T \qquad (3)$$

The value of the integral is given by the c_0 term alone. All other terms in the expansion integrate to zero.

Let $h = T/N; \quad t_k = h\left(k - \frac{1d0}{2}\right)$ so that the midpoint trap rule approximation to this integral is

$$\int_0^T f(t)dt = \sum_{j=0}^{\infty} c_j h \sum_{k=1}^{k=N} e^{2\pi i j(t_k/T)} = \frac{T}{N} c_0 \sum_{k=1}^{N} 1 + \frac{T}{N} \sum_{j=1}^{\infty} c_j \sum_{k=1}^{k=N} e^{2\pi i j \frac{k-1d0/2}{N}} \qquad = c_0 T + \frac{T}{N} \sum_{j=1}^{\infty} c_j e^{-\pi i j} \sum_{k=1}^{k=N} e^{2\pi i j \frac{k}{N}}$$

(4)

Note the interchange of summation orders. Let $z_j = e^{2\pi i \frac{j}{N}}$, so that

$$\int_0^T f(t)dt - c_0 T = \frac{T}{N} \sum_{j=1}^{\infty} c_j e^{-\pi i j} \sum_{k=1}^{k=N} z_j^{\,k} \qquad (5)$$

The last term can be written as a sum of z^k and we can use the familiar relation for $z \neq 1$

$$\frac{1}{1-z_j} = \sum_{k=0}^{\infty} z_j^{\,k} = \sum_{k=0}^{N-1} z_j^{\,k} + \sum_{k=N}^{\infty} z_j^{\,k} = \sum_{k=0}^{N-1} z_j^{\,k} + \frac{z_j^{\,N}}{1-z_j} \qquad (6)$$

So that $$\sum_{k=0}^{N-1} z_j^{\,k} = \frac{z_j^{\,N} - 1}{1 - z_j} \qquad (7)$$

Note that $z_j^{\,N} = e^{2\pi i \frac{jN}{N}} = e^{2\pi i j} = 1$ so that the numerator in (7) is always zero. The denominator is also zero for j=mN for which $z_{mN} = e^{2\pi i \frac{mN}{N}} = 1$. In this case the last sum in (5) is N so that

$$\int_0^T f(t)dt - c_0 T = T \sum_{m=1}^{\infty} c_{mN} e^{-\pi i m N} \qquad (8)$$

The terms that enter are not uniform. If for some accidental reason $c_{10} = 0$, 10 point integration will seem exact up to c_{20}. Changing N from 10 to 20 will give a false accuracy reading. The correct change is from 10 to 11 or 13, **Do not simply double the points.**